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TOPICAL ISSUE

Maximum Achievable Precision of Linear Systems with Discrete Controllers

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Abstract—Consideration was given to the problem of maximum achievable precision of linear systems with discrete state and output controllers. The external perturbations affecting the system are the bounded step-type and harmonic (of unknown frequency) vector functions of time which the control theory regards as standard. The condition for asymptotic stability of the closed-loop system is the only requirement on the controllers aside from their physical realizability. Therefore, the conclusions of the present paper apply to the entire set of the discrete stabilizing controllers, no matter what method was used to design them.

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1. INTRODUCTION

For the continuous systems, there are control laws providing an arbitrarily high control precision. This is possible, for example, if the external perturbations and control actions are applied to the same point and the control law is designed from the full state vector on the basis of the LQ procedures [1] of the H_∞-optimization [2]. As follows from [3–5], these systems admit an unbounded increase in the controller gains without loss of asymptotic stability of the closed-loop system, which enables an arbitrarily high control precision.

For the discrete systems, a high transfer coefficient defining precision of the open-loop system cannot be provided without loss in stability of the closed-loop system. For the single-input single-output systems of first and second orders, this was noticed by Ya.Z. Tsypkin [6]. It is due to the fact that the Nyquist plot of the discrete-time systems intersects without fail the segment $[-1; 0]$ of the real axis, which is indicative of impossibility of the unbounded increase in the transfer coefficient of the open-loop system without loss in stability.

As follows from [4, 7, 8], in distinction to the continuous case, the discrete counterparts of the procedures of LQ-optimization do not admit an unbounded increase in the controller gains without loss in the stability of the closed-loop system. It is indicative of the fundamental impossibility of providing an arbitrarily high control precision in the systems with discrete-time controllers. The present paper is devoted to studying this phenomenon.

Bounded precision of the systems with discrete-time controllers is the fundamental distinction of the discrete-time systems (with time quantization) as compared with the continuous systems. This is explained by the specificity of their modal stability domain having the form of the unit circle. The lower the system order and the greater the period of quantization of the discrete-time controller, the stronger the effect of this fundamental constraint.

Consequently, at using the discrete-time control one can imply only some maximum control precision achievable for the given period of quantization of the discrete-time controller. Attention was drawn to this fact for the first time in [9]. For the step-type external perturbations, some results were obtained in 1994 [10] also for the output controllers.

The present paper considers this problem for the arbitrary-order plants both in the one-dimensional and multidimensional cases. Generally speaking, in the continuous case the maximum precision was first considered in [11, 12] from the standpoint of the limiting value of the quadratic functional at parrying arbitrary initial conditions, rather than suppression of the external perturbations. Later on A.A. Pervozvanskii and M.A. Pasumanskii considered this issue from the positions of reducing the impact of the external perturbations in the sense of the limiting value H_2 and H_∞-norm [13]. We notice, however, that the results of [13] almost repeat the conclusions of [11, 12], that is, come to the fact that the plant must be minimal phase with the same number of controls and measurements coinciding with the controlled variables, and disregard the point of application of the external perturbations and control actions, which is extremely important [5, 14]. The problem of precision of the continuous and discrete-time systems was considered at the same time internationally in [15–17].

The present paper proposes in essence a different approach to the problem of precision relying on the estimation of the maximal possible transfer coefficient of the open-loop system (or the value of the recurrent difference at the frequency of the external perturbation) which defines the precision characteristics of the closed-loop system. The study is carried out both for the one-dimensional (scalar) control and the multidimensional (vector) control.

It deserves noting that the vanishing limiting error of the weight coefficient under the control in quadratic functional (the case of "inexpensive" control) was established in [18] after the appearance of [9, 10] devoted to the maximal transfer coefficient of the discrete-time state LQ-controllers. The same result was established in [19] for the output controllers based on the reduced-order Luenberger observer. We notice that constraint of the precision of discrete-time systems follows immediately from the problem of l_1-optimal control [20–23].

2. FULL STATE VECTOR CONTROLLERS

2.1. Formulation of the Problem

Let us consider a fully controllable plant with the discrete model given by

$$\begin{gathered} x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \quad x \in R^n, \quad u \in R^m, \quad w \in R^\mu, \\ z(k) = Cx(k), \quad k = 0, 1, 2, \ldots, \quad z \in R^{m_1}, \end{gathered} \tag{2.1}$$

where x is the measurable plant state vector; u is the control action vector; w is the perturbation vector; z is the vector of plant controllable variables; and A, B_1, B_2, C are certain matrices of the corresponding sizes.

We confine consideration to the case of coinciding points of application of the control and perturbing actions, the number of the control actions and controlled variables being the same:

$$B_1 = B_2, \quad m = m_1 = \mu. \tag{2.2}$$

In the continuous case, the last condition ensures knowingly existence of the full state-vector controller providing an arbitrarily high control precision [1, 5]. This allows one to characterize more prominently the precision of the discrete-time controllers as compared with the continuous ones.

We embrace plant (2.1) by a discrete-time state controller

$$u(k) = Kx(k) \tag{2.3}$$

with parameters established by any of the existing methods providing asymptotic stability of the closed-loop system (2.1), (2.3) for $w = 0$.

We establish relation between the vectors of controlled variables and perturbing actions in the closed-loop system (2.1), (2.3). Since for the controlled plant $W_0(q) = C(qI - A)^{-1}B_2$, q being the operator of time shift ahead by one step, the matrix of transfer from u to z is independent of the controller parameters, the properties of the closed-loop system and, in particular, its precision characteristics are wholly defined by the frequency properties of the recurrent difference matrix [5, 7]:

$$V(q) = I_m + W_p(q). \tag{2.4}$$

Here, $W_p = -K(qI - A)^{-1}B_2$ is the transfer matrix of the open-loop system (2.1), (2.3) in plant inputs (variable u) and I_m is the $(m \times m)$ identity matrix.

The inverse matrix of $V(q)$ is involved in the expression of the operator transfer matrix (from w to z) of the closed-loop system [5, 7]:

$$T_{zw}(q) = W_0(q)[I_m + W_p(q)]^{-1}, \quad z(k) = T_{zw}(q)w(k). \tag{2.5}$$

The present paper aims at studying the frequency characteristics of the recurrent difference matrix (2.4) in system (2.1), (2.2) with the discrete-time state controller (2.3) and estimating the maximum achievable values of the steady-state control errors under the action of standard bounded step-type and harmonic external perturbations. Prior to considering the general case of vector control and vector controllable variable, we consider the scalar case.

2.2. Scalar Control

Let us consider the case of single control action and single controlled variable $m = m_1 = \mu = 1$. As follows from (2.5), the expression for the controlled variable is given by

$$z(k) = \frac{w_0(q)}{1 + w_p(q)}\, w(k), \tag{2.6}$$

where w_0, w_p are the transfer functions of the controlled plant and the open-loop system, respectively; $w(k)$ and $z(k)$ are the external perturbation and the controlled variable at the current kth instant of the discrete time.

We assume that the external perturbation is constant $w(k) = \text{const} = w^*$, where $w^* > 0$ is the perturbation amplitude. According to the theorem of limiting values [12, 17, 24, 25], for the steady value of the controlled variable (control error) it then follows from (2.6) that

$$\lim_{k\to\infty} z(k) = z(\infty) = \frac{w_0(1)}{1 + w_p(1)}\, w^* = \frac{k_0}{1 + k_p}\, w^*, \tag{2.7}$$

where, as is known from [7, 12, 17, 24–26],

$$k_0 = w_0(q)_{|q=1} = w_0(1) \quad \text{and} \quad k_p = w_p(q)_{|q=1} = w_p(1) \tag{2.8}$$

are called the transfer coefficients (gains), respectively, of the controlled plant and open-loop discrete-time system. One can see from (2.7) that the greater the transfer coefficient of the open loop k_p, the smaller the steady-state error of control because the numerator (2.7) is independent of the controller coefficients and is defined only by the plant properties. In the discrete case, however, this transfer coefficient is in essence bounded from above, which gives rise to the need for determining the maximum achievable control error in the systems with the discrete-time controller. This fact is established by the following theorem.

Theorem 1. *Under a constant external perturbation $w(k) = \text{const} = w^* > 0$, the transfer coefficient of the open-loop discrete-time system* (2.1), (2.3) *with scalar control and the maximum achievable control error satisfy the inequalities*

$$|1 + k_p| < \frac{2^n}{|d(1)|}, \quad d(1) = d(q)\Big|_{q=1} \neq 0, \quad d(q) = \det(qI_n - A), \tag{2.9}$$

$$|z(\infty)| > \frac{|k_0|\ |d(1)|}{2^n}\ w^*. \tag{2.10}$$

Theorem 1 is proved in the Appendix.

Remark 1. Inequality (2.9) makes sense if the characteristic polynomial of the open-loop system $d(q) = \det(qI_n - A)$ has no roots at the point $q = 1$ (the plant has no integrating units). Otherwise, this polynomial is representable as $d(q) = (q-1)^l d_1(q)$, where l is the multiplicity of the root $q = 1$ and the polynomial $d_1(q)$ has no roots at the point $q = 1$. In this case, the transfer coefficient of the plant and the gain of the open-loop system are given by

$$k_0 = [(q-1)^l w_0(q)]\Big|_{q=1}, \quad k_p = [(q-1)^l w_p(q)]\Big|_{q=1}. \tag{2.11}$$

By multiplying the numerator and denominator in the right side of (2.6) by $(q-1)^l$ we then obtain for the steady-state control error that

$$z(\infty) = \frac{(q-1)^l w_0(q)}{(q-1)^l + (q-1)^l w_p(q)}\Bigg|_{q=1} w^* = \frac{k_0}{k_p}\ w^*. \tag{2.12}$$

Having first multiplied both sides of the identity (A.1) by $(q-1)^l$), similar to the proof of Theorem 1 one can demonstrate that the transfer coefficient of the open-loop system and the maximum achievable control error satisfy the inequalities

$$|k_p| < \frac{2^n}{|d_1(1)|}, \quad d_1(1) \neq 0, \quad |z(\infty)| > \frac{|k_0|\ |d_1(1)|}{2^n}\ w^*. \tag{2.13}$$

Now we assume that the external perturbation is a harmonic lattice function like $w(k) = w^* \sin(\omega kT)$, where $w^* > 0$ is the perturbation amplitude, ω is the circular frequency (unknown as a rule), and T is the period of discreteness (quantization) of the discrete-time controller (2.3). According to the classical theory of linear discrete-time systems [12, 17, 24, 25], the amplitude $a > 0$ of the steady-state output oscillations of the controlled variable

$$\lim_{k\to\infty} z(k) = a \sin(\omega kT + \phi), \quad k = 0, 1, 2, \ldots \tag{2.14}$$

(ϕ is the phase shift relative to the input harmonic) can be established from

$$a = |T_{zw}(e^{j\omega T})| w^* = \frac{|w_0(e^{j\omega T})|}{|1 + w_p(e^{j\omega T})|}\ w^*, \tag{2.15}$$

where $T_{zw}(q)$ is the transfer function of the closed-loop system from (2.6).

The denominator in (2.15) includes the magnitude of the recurrent difference calculated at the frequency ω of the external perturbation. The greater this magnitude, the smaller the amplitude a of forced oscillations of the controlled variable z. However, as is established in Theorem 2, in the discrete case this value is always bounded from above and, consequently, there exists a maximum achievable amplitude of oscillations of the controlled variable which in essence cannot by reduced by the state controller (2.3).

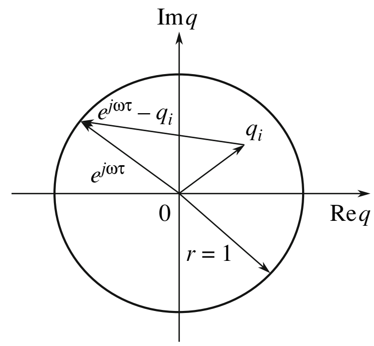

Figure.

Theorem 2. *Under the harmonic external perturbation* $w(k) = w^* \sin(\omega kT)$, *the recurrent difference of the discrete-time system* (2.1), (2.3) *with scalar control and the maximum achievable amplitude of the control error satisfy the inequalities*

$$|1 + w_p(e^{j\omega T})| < \frac{2^n}{|d(e^{j\omega T})|}, \quad d(e^{j\omega T}) \neq 0, \tag{2.16}$$

$$a > \frac{|w_0(e^{j\omega T})|\ |d(e^{j\omega T})|}{2^n}\ w^* \tag{2.17}$$

for all real frequencies $\omega \in [0, \pi/T]$.

Theorem 2 is proved in the Appendix.

Remark 2. Theorem 1 follows from Theorem 2 under the zero frequency of the external perturbation $\omega = 0$. One can also conclude from (2.9) and (2.16) that the greater the order of the plant n, the greater the magnitudes of the maximum transfer coefficient of the open-loop discrete-time system (recurrent difference at the zero frequency) and the recurrent difference calculated at the frequency of the external perturbation. If the roots of the characteristic polynomial $D(q)$ of the closed-loop system (2.1), (2.3) approach the boundary of the unit circle, then these values are maximized, which leads to minimization of the steady-state control error. In particular, if $q_i \to -1$, $i = \overline{1, n}$, then the transfer coefficient of the open-loop system is maximized. If $q_i \to -e^{-j\omega T}$, $i = \overline{1, n}$, the magnitude of the recurrent difference at the frequency of external perturbation is maximized. One can easily make sure of this fact from the geometric interpretation of the magnitude of the ith multiplier $|e^{j\omega T} - q_i|$ of the expansion $|D(e^{j\omega T})|$ shown in the figure. However, assignment of such roots of the polynomial $D(q)$ (for example, by means of the procedure of modal control) leads, obviously, to prolonged and oscillatory transients in the closed-loop system.

2.3. Vector Control

As before, we assume that condition (2.2) is satisfied and consider first the case of constant external perturbations $w(k) = \text{const} = w^*$, where $w^* = [w_1^*, w_2^*, \ldots, w_m^*]^{\mathrm{T}}$ is the vector of perturbation amplitudes and T denotes the matrix transposition. Then, the vector of steady-state control errors is established from the following equality [12, 17, 24, 25]:

$$\lim_{k\to\infty} z(k) = z(\infty) = W_0(1)[I_m + W_p(1)]^{-1} w^*. \tag{2.18}$$

By assuming that $q = 1$ is not a root of the numerator $\det W_0(q)$, that is, zero of the plant and the plant is statically definable [14] because, otherwise, there will be an infinite number of vectors $w^* \neq 0$ such that $z(\infty) = 0$, we express from (2.18) the vector w^* for $m_1 = m = \mu,\ B_1 = B_2$ as

$$[I_m + W_p(1)]\ {W_0}^{-1}(1) z(\infty) = w^*$$

and generate the quadratic form

$$z^{\mathrm{T}}(\infty)[{W_0}^{-1}(1)]^{\mathrm{T}}\,[I_m + W_p(1)]^{\mathrm{T}}\,[I_m + W_p(1)]\;{W_0}^{-1}(1)z(\infty) = (w^*)^{\mathrm{T}}\,w^*. \tag{2.19}$$

We notice that the control law (2.3) defines the matrix of recurrent difference $V(q) = I_m + W_p(q)$ that appears in the left side of the latter equality and is calculated for $q = 1,\ q = e^{j\omega T},\ \omega = 0$ at the zero frequency. It follows from (2.19) that the values of the static errors depend on the positive definite symmetrical matrix $V^{\mathrm{T}}(1)V(1)$ representable as [27]

$$V^{\mathrm{T}}V = U\ \mathrm{diag}({\sigma_1}^2, {\sigma_2}^2, \ldots, {\sigma_m}^2)\,U^{\mathrm{T}},$$

where U is the orthogonal matrix ($U^{\mathrm{T}}U = UU^{\mathrm{T}} = I_m$) and ${\sigma_i}^2 > 0$ ($i = \overline{1, m}$) are the eigenvalues of the matrix $V^{\mathrm{T}}V$ (squares of the singular values of the matrix V).

By denoting

$$\tilde{z} = {W_0}^{-1}z(\infty), \quad \tilde{y} = U^{\mathrm{T}}\,\tilde{z},$$

we rearrange the quadratic form (2.19) as

$$\tilde{y}^{\mathrm{T}}\ \mathrm{diag}({\sigma_1}^2, {\sigma_2}^2, \ldots, {\sigma_m}^2)\,\tilde{y} = (w^*)^{\mathrm{T}}\,w^*. \tag{2.20}$$

We notice that in virtue of orthogonality of the matrix U the vectors $\tilde{z}$ and $\tilde{y}$ have the same Euclidean norm because their scalar products are equal [27]:

$$\|\tilde{y}\|^2 = \tilde{y}^{\mathrm{T}}\tilde{y} = [U^{\mathrm{T}}\tilde{z}]^{\mathrm{T}}\,U^{\mathrm{T}}\tilde{z} = \tilde{z}^{\mathrm{T}}U\ U^{\mathrm{T}}\tilde{z} = \tilde{z}^{\mathrm{T}}\tilde{z} = \|\tilde{z}\|^2.$$

At that, the greater the norm of the vector $z(\infty)$ of static errors, the greater the norm of the vector $\tilde{z}$, and, otherwise, because

$$\tilde{z}^{\mathrm{T}}\tilde{z} = z^{\mathrm{T}}(\infty)[{W_0}^{-1}(1)]^{\mathrm{T}}\,{W_0}^{-1}(1)\ z(\infty),$$

and the matrix $[{W_0}^{-1}(1)]^{\mathrm{T}}\,{W_0}^{-1}(1)$ is symmetrical and positive definite.

Rearrange (2.20) in the scalar form as

$$\sum_{i=1}^{m} \tilde{y}_i^2{\sigma_i}^2 = \sum_{i=1}^{m}(w_i^*)^2 \tag{2.21}$$

from which it follows that the greater the minimal singular value of the matrix of recurrent difference $V(1)$, the smaller the norm of the vector $\tilde{y}$ which is equal to the norm of the vector $\tilde{z}$, and, consequently, the smaller the norm of the static errors $z(\infty)$. In particular, this corresponds to the well-known qualitative results for the multidimensional continuous systems [26, 28].

However, as will be established in Theorem 3, for the systems with discrete-time controllers the minimal singular value $\sigma_{\min}$ of the matrix of recurrent difference $V(1)$ is in essence bounded from above. As in the one-dimensional case, it is, therefore, possible to discuss the maximum achievable precision of control estimated by the norm of the vector of steady-state control errors. In the multidimensional case, this fact can be interpreted geometrically. Equality (2.19) implies that the representing point of the vector of static control errors $z(\infty) \in R^m$ lies on the surface of the hyperellipsoid defined by this equality. Its volume V_m in the m-dimensional space characterizes indirectly the "value" of the vector $z(\infty)$. Bounded precision of the discrete-time systems over the entire set of the stabilizing state controllers (2.3) means that it is impossible in principle to obtain ellipsoid (2.19) of volume smaller than some limiting volume defined only by the properties of the controlled plant and the norm of the external perturbation. These considerations are formulated in the mathematical terms in the following theorem.

Theorem 3. *The minimal singular value of the matrix of recurrent difference* $V(q) = I_m + W_p(q)$ *at the zero frequency* ($q = e^{j\omega T}$, $\omega = 0$, $q = 1$) *and the volume* V_m *of the hyperellipsoid* (2.19) *whose surfaces belong to the representing point of the vector of static control errors* $z(\infty) \in R^m$ *of the discrete-time multidimensional system* (2.1), (2.3) *under a constant external perturbation satisfy the inequalities*

$$\sigma_{\min}[I_m + W_p(1)] < \left[\frac{2^n}{|d(1)|}\right]^{1/m}, \quad d(1) \neq 0, \tag{2.22}$$

$$V_m > \frac{\pi^{m/2}}{\Gamma(m/2+1)} \frac{|\det W_0(1)||d(1)|}{2^n} \|w^*\|^m, \tag{2.23}$$

where $\sigma_{\min}[V]$ *is the minimal singular value of the matrix* V *and* $\Gamma(b)$ *is the gamma function of the variable* b.

Theorem 3 is proved in the Appendix.

It deserves noting that Theorem 3 gives no explicit estimates of the norm of the vector of steady-state errors. Determination of these errors is the subject matter of further discussion. For that, in virtue of (2.18) we set down that

$$\|z(\infty)\|^2 = w^{*\mathrm{T}}([I_m + W_p(1)]^{-1})^{\mathrm{T}}\, {W_0}^{\mathrm{T}}(1)\; W_0(1)[I_m + W_p(1)]^{-1} w^* \tag{2.24}$$

and transform the right side of (2.24). Since the number matrix ${W_0}^{\mathrm{T}}(1)\; W_0(1)$ is a positive definite one and its eigenvalues are the squares of the singular values of the matrix $W_0(1)$, the inequalities

$$\sigma^2_{\min}[W_0(1)]\; f^{\mathrm{T}} f \leqslant f^{\mathrm{T}} W_0^{\mathrm{T}}(1)\; W_0(1)\; f \leqslant \sigma^2_{\max}[W_0(1)]\; f^{\mathrm{T}} f$$

will be valid for all nonzero vectors $f \in R^m$ [27]. With regard for these inequalities, it is possible to establish from (2.24) that

$$\|z(\infty)\|^2 \leqslant \sigma^2_{\max}[W_0(1)]\; f^{\mathrm{T}} f, \quad \|z(\infty)\|^2 \geqslant \sigma^2_{\min}[W_0(1)]\; f^{\mathrm{T}} f, \tag{2.25}$$

where $f = [I_m + W_p(1)]^{-1} w^*$ and $\sigma_{\max}[V]$ is the maximal singular value of the matrix V.

Now we consider the quadratic form

$$f^{\mathrm{T}} f = w^{*\mathrm{T}}([I_m + W_p(1)][I_m + W_p(1)]^{\mathrm{T}})^{-1} w^*.$$

Since $[I_m + W_p(1)][I_m + W_p(1)]^{\mathrm{T}}$ is a positive definite matrix and its eigenvalues are the squares of the singular values of the matrix $V(1) = [I_m + W_p(1)]$, we can obtain the inequalities

$$f^{\mathrm{T}} f \leqslant \frac{1}{\sigma^2_{\min}[I_m + W_p(1)]}\; w^{*\mathrm{T}} w^*, \quad f^{\mathrm{T}} f \geqslant \frac{1}{\sigma^2_{\max}[I_m + W_p(1)]}\; w^{*\mathrm{T}} w^*$$

with allowance for the fact that at inversion of a matrix its eigenvalues are inverted as well [27]. By substituting these inequalities in (2.25), we finally establish that the steady-state errors of the multidimensional discrete-time system (2.1), (2.3) satisfy the inequalities

$$\frac{\sigma_{\min}[W_0(1)]}{\sigma_{\max}[I_m + W_p(1)]}\; \|w^*\| \leqslant \|z(\infty)\| \leqslant \frac{\sigma_{\max}[W_0(1)]}{\sigma_{\min}[I_m + W_p(1)]}\; \|w^*\|. \tag{2.26}$$

We notice that, as follows from inequalities (A.7), in these inequalities there are bounded numbers to the left and right of the denominator. Consequently, in the multidimensional discrete-time systems the lower boundary of the norm of the vector of static control errors is always finite.

We consider the case of harmonic external perturbations

$$w(k) = w^* \sin(\omega k T), \tag{2.27}$$

where $w^* \in R^m$ is the vector of amplitudes of the external perturbation, ω is the circular—generally, unknown—frequency, and T is the discreteness (quantization) period of the discrete-time controller (2.3).

According to the results of the theory of multidimensional linear discrete-time systems [12, 15, 25], the output steady-state oscillations in each of the controlled variable (control errors) are given by

$$\lim_{k\to\infty} z_i(k) = a_i \sin(\omega k T + \phi_i), \quad i = \overline{1, m}, \tag{2.28}$$

where a_i and ϕ_i are, respectively, the amplitude and phase shift of the ith output harmonic relative to the input perturbation (2.27).

We notice that the amplitudes of oscillations in each of the controlled variables are the magnitudes of the corresponding components of the following complex conjugate vectors [29]:

$$z_+ = T_{zw}(e^{j\omega T}) w^* e^{j\omega k T}, \quad z_- = T_{zw}(e^{-j\omega T}) w^* e^{-j\omega k T},$$

where z_+ and z_- are partial solutions of the system of difference Eqs. (2.1), (2.3), respectively, for $w(k) = w_+(k) = w^* e^{j\omega k T}$ and $w(k) = w_-(k) = w^* e^{-j\omega k T}$ [25], and $T_{zw}(e^{j\omega T})$ is the frequency transfer function of the closed-loop discrete-time system (2.5). Indeed, one can readily see that the input vector (2.27) is representable as $w(k) = (w_+(k) - w_-(k))/(2j)$, and one can put down $(z_+ - z_-)/(2j)$ for the output vector z with the components from (2.28) in virtue of the principle of superposition. Therefore, we conclude that $a_i^2 = z_{-i} z_{+i}$, where z_{-i} and z_{+i} are, respectively, the ith components of the vectors z_- and z_+. Or else, if defined is the vector $a = [a_1, a_2, \ldots, a_m]^{\mathrm{T}}$ whose coordinates the amplitudes of the harmonics (2.28), then obviously $a^{\mathrm{T}} a = \|a\|^2 = z_-^{\mathrm{T}} z_+$.

With regard for (2.5), the last equality is representable as

$$\|a\|^2 = w^{*\mathrm{T}} ([I_m + W_p(e^{-j\omega T})]^{\mathrm{T}})^{-1} W_0{}^{\mathrm{T}}(e^{-j\omega T})\ W_0(e^{j\omega T}) [I_m + W_p(e^{j\omega T})]^{-1} w^*. \tag{2.29}$$

It is a counterpart of equality (2.24). Therefore, by similar reasoning—to within the replacement of the real vectors by the complex ones and the positive definite matrices by the corresponding Hermitian matrices—we obtain an evident analog of (2.26):

$$\frac{\sigma_{\min}[W_0(e^{j\omega T})]}{\sigma_{\max}[I_m + W_p(e^{j\omega T})]}\ \|w^*\| \leqslant \|a\| \leqslant \frac{\sigma_{\max}[W_0(e^{j\omega T})]}{\sigma_{\min}[I_m + W_p(e^{j\omega T})]}\ \|w^*\| \tag{2.30}$$

bounding the Euclidean norm of the vector of amplitudes of the steady-state oscillations from (2.28).

It is shown below that the denominator in these inequalities has bounded numbers to its left and right. Therefore, the lower boundary of the norm of the vector of amplitudes of the control errors in the multidimensional discrete-time systems is always finite. This fact can be interpreted geometrically as that in Theorem 3.

By assuming that the value of $q = e^{j\omega T}$ is not a root of the numerator $\det W_0(q)$, that is, the plant zero because, otherwise, there will be an infinite number of vectors $w^* \neq 0$ from (2.27) such that $a = 0$, we generate an analog of the quadratic form (2.19), the Hermitian form

$$z_-^{\mathrm{T}} [W_0{}^{-1}(e^{-j\omega T})]^{\mathrm{T}}\, [I_m + W_p(e^{-j\omega T})]^{\mathrm{T}}\, [I_m + W_p(e^{j\omega T})]\ W_0{}^{-1}(e^{j\omega T}) z_+ = (w^*)^{\mathrm{T}} w^*$$

rearranged equivalently in

$$(x - jy)^{\mathrm{T}} Q(x + jy) = 1, \tag{2.31}$$

where $z_+ = x + jy$, $z_- = x - jy$ ($x, y \in R^m$ are real vectors) and Q is a Hermitian matrix like

$$Q = [{W_0}^{-1}(e^{-j\omega T})]^{\mathrm{T}} \, [I_m + W_p(e^{-j\omega T})]^{\mathrm{T}} \, [I_m + W_p(e^{j\omega T})] \; {W_0}^{-1}(e^{j\omega T})/\|w^*\|^2.$$

The left side of the Hermitian form (2.31) includes the complex vectors and matrix. Nevertheless, this Hermitian form which assumes, as is well known, real values can be rearranged in the real quadratic form of $2m$ variables [27]:

$$\begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x^{\mathrm{T}} \quad y^{\mathrm{T}}] M [x^{\mathrm{T}} \quad y^{\mathrm{T}}]^{\mathrm{T}} = 1, \tag{2.32}$$

where A and B are the real and imaginary parts of the matrix $Q = A + jB$ which in virtue of the Hermitian character of Q are the real symmetric, $A^{\mathrm{T}} = A$, and skew-symmetric, $B^{\mathrm{T}} = -B$, matrices. In particular, this implies that the matrix M of the given quadratic form is real symmetric and its determinant $\det M = \det Q \det \overline{Q} = |\det Q|^2$ [27], where $\overline{Q} = A - jB$. We notice that the norm of the composite vector $[x^{\mathrm{T}} \quad y^{\mathrm{T}}]^{\mathrm{T}}$ is equal to the norm of the vector of oscillation amplitudes a. Therefore, the volume V_{2m} of hyperellipsoid (2.32) in the $2m$-dimensional space characterizes indirectly the "values" of the vector a. The fact that the discrete-time systems have bounded precision over the entire set of the stabilizing state controllers (2.3) implies that in principle it is impossible to obtain an ellipsoid (2.32) of a volume smaller than some limiting volume defined only by the properties of the controlled plant and the norm of external perturbation. The following theorem formulates these considerations in the mathematical terms.

Theorem 4. *The minimal singular value of the matrix of recurrent difference* $V(q) = I_m + W_p(q)$ *calculated at the frequency of external perturbation* (*for* $q = e^{j\omega T}$), *and the volume* V_{2m} *of hyperellipsoid* (2.32) *characterizing indirectly the "value" of the vector of amplitudes of the control errors* $a \in R^m$ *of the discrete-time multidimensional system* (2.1), (2.3) *under the action of harmonic external perturbations* (2.27) *for all real frequencies* $\omega \in [0, \pi/T]$ *obey the inequalities*

$$\sigma_{\min}[I_m + W_p(e^{j\omega T})] < \left[\frac{2^n}{|d(e^{j\omega T})|}\right]^{1/m}, \quad d(e^{j\omega T}) \neq 0,$$

$$V_{2m} > \frac{\pi^m}{m!} \; \frac{|\det W_0(e^{j\omega T})|^2 \, |d(e^{j\omega T})|^2}{2^{2n}} \; \|w^*\|^{2m}. \tag{2.33}$$

Theorem 4 is proved in the Appendix.

3. OUTPUT CONTROLLERS

Since the systems with discrete-time controllers have bounded precision even in the case of full measurement of the system state vector, it is understandable intuitively that this effect is all the more possible under the output control. It follows from [1] that for the minimal-phase plants where the measured variables coincide with the controlled ones and their number is equal to the dimensionality of control there exists knowingly a continuous output control law providing an arbitrarily high control precision (stabilization). Therefore, in what follows we confine our consideration namely to this class of plants, which enables us to specify more clearly the distinctions of the precision characteristics of the continuous and discrete-time control systems and compare them. Additionally, for the sake of simplicity we confine ourselves to the case of scalar control action and controlled variable because as the last section demonstrated in essence the multidimensional case does not introduce any new facts but is more cumbersome. The multidimensional case can be considered by analogy with Section 2.

3.1. Formulation of the Problem

Let us consider a discrete model of the continuous plant given in the "input-output" form by the equations

$$d(q)y(k) = b(q)u(k) + l(q)w(k), \quad y \in R^1, \quad u \in R^1, \quad w \in R^1, \tag{3.1}$$

with the discrete-time output controller

$$g(q)u(k) = r(q)y(k), \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where y is the controllable and concurrently measurable variable of the plant; u is the control action; w is the external perturbation; and $d(q), b(q), l(q), g(q), r(q)$ are certain polynomials of the operator q. The higher degree of the polynomial $d(q)$ exceeds by one the higher degrees of the polynomials $b(q)$ and $l(q)$, and the higher degree of the polynomial $r(q)$ does not exceed the higher degree of the polynomial $g(q)$ (the controller is physically realizable).

We assume that the plant is fully controllable and observable, that is, the polynomials $d(q)$ and $b(q)$ have no common roots, and the controller determined by any existing method is such that for $w = 0$ the closed-loop system (3.1), (3.2) is asymptotically stable.

We determine the following transfer functions:

$$w_0(q) = \frac{b(q)}{d(q)}, \quad w_f(q) = \frac{l(q)}{d(q)}, \quad w_r(q) = \frac{r(q)}{g(q)}, \quad w_p(q) = -w_0(q)w_r(q), \tag{3.3}$$

where w_0 is the plant transfer function in the control action; w_f is the plant transfer function in the perturbing action; w_r is the controller transfer function; and w_p is the transfer function of the open-loop system (3.1), (3.2) in the physical output (input) of the plant.

By considering together (3.1), (3.2), and (3.3), we establish the relation between the controlled variable and the perturbing action

$$y(q) = \frac{w_f(q)}{1 + w_p(q)}\, w(q) = \frac{g(q)l(q)}{d(q)g(q) - r(q)b(q)}\, w(q), \tag{3.4}$$

where the denominator includes the characteristic polynomial of the closed-loop system (3.1), (3.2)

$$D(q) = d(q)g(q) - r(q)b(q)$$

all of whose roots q_i satisfy the conditions $|q_i| < 1$ (A.3) in virtue of asymptotic stability. Without loss of generality one can assume that the coefficient at the higher degree of q of this polynomial is equal to one.

Since the plant transfer function in perturbation is independent of the controller parameters, it follows from (3.4) that the properties of the closed-loop system and, in particular, its precision characteristics are defined wholly by the frequency properties of the recurrent difference $v(q) = 1 + w_p(q)$ which is the denominator of the transfer function from w to y of the closed-loop system

$$T_{yw}(q) = \frac{w_f(q)}{1 + w_p(q)} = \frac{g(q)l(q)}{d(q)g(q) - r(q)b(q)}.$$

We consider the frequency properties of the recurrent difference $v(q)$ in system (3.1), (3.2) with the discrete-time output controller and estimate the maximum achievable values of the steady-state control errors in the case where the plant is subjected to the standard bounded step-type and harmonic external perturbations.

3.2. Step-type Perturbations

We assume that the external perturbation is constant $w(k) = \text{const} = w^*$, where $w^* > 0$ is the perturbation amplitude. According to the theorem about the limiting values [24, 25], for the steady-state value of the controlled variable (control error) we obtain from (3.4) that

$$\lim_{k\to\infty} y(k) = y(\infty) = \frac{w_f(1)}{1 + w_p(1)}\, w^* = \frac{k_f}{1 + k_p}\, w^* = \frac{g(1)l(1)}{D(1)}\, w^*, \tag{3.5}$$

where $k_f = w_f(1)$ and $k_p = w_p(1)$ are the perturbation transfer coefficients (gains) of the plant and the open-loop discrete-time system, respectively. As can be seen from (3.5), the higher the transfer coefficient of the open loop, the lower the steady-state control error. However, as it was in the case of discrete-time state controllers, this transfer coefficient is in essence bounded from above, which leads to the need for indicating the maximum achievable control error in the systems with discrete-time output controller. This fact looks most natural in the minimal-phase case because, otherwise, the precision of even continuous systems is bounded. That is why we assume in what follows that the polynomial of the plant transfer function numerator in the control action $b(q)$ has roots within the unit circle $|q| < 1$.

We notice that the recurrent difference $v(q)$ satisfies the following identity which is similar to (A.1) and easily verifiable:

$$1 + w_p(q) = \frac{D(q)}{d_p(q)}, \tag{3.6}$$

where $d_p(q) = d(q)g(q)$ is the characteristic polynomial of the open-loop system. Assuming that $q = 1$ in (3.6) and taking into consideration that the inequality (A.4), which is proved in the Appendix, is satisfied in virtue of stability of the polynomial $D(q)$, we obtain the inequality

$$|1 + k_p| < \frac{2^n}{|d_p(1)|}, \quad d_p(1) = d(1)g(1) \neq 0, \tag{3.7}$$

where n is the degree of the polynomial $D(q)$.

We note that since there is a finite number in the denominator of (3.7), the transfer coefficient of the open-loop discrete-time system is always bounded from above. In turn, the maximum achievable static condition of control satisfies the inequality following from (3.5):

$$|y(\infty)| > \frac{|k_f|\,|d_p(1)|}{2^n}\, w^* = \frac{|g(1)|\,|l(1)|}{2^n}\, w^*, \quad d_p(1) = d(1)g(1) \neq 0. \tag{3.8}$$

If the plant and/or controller have the integrating units ($d_p(1) = d(1)g(1) = 0$), then the above inequalities can be modified by analogy with Remark 1.

We emphasize that the resulting inequalities and conclusions are valid independently of stability and/or minimal phase of the plant. However, they have an essential distinction from the similar inequalities of Theorem 1. The right side of these inequalities includes the characteristic polynomial of the plant $d(q)$ and the characteristic polynomial $g(q)$ of the controller as calculated at the zero frequency for $q = 1$. At the same time, if the plant is a minimal phase one, then many methods of design of the discrete-time output controller [7, 20] and, in particular, the optimal control l_1-theory [21–23] provide controllers with $g(q) = b(q)$, which means that the zeros of the plant lying within the unit circle are compensated by the controller poles. In this case, the polynomial $b(q)$ is the multiplier of the characteristic polynomial of the closed-loop system $D(q) = b(q)[d(q) - r(q)]$, and the above inequalities obtained by cancelling the polynomial $b(q)$ in (3.4) and (3.6)) are given by

$$|1 + k_p| < \frac{2^\rho}{|d(1)|}, \quad |y(\infty)| > \frac{|l(1)|}{2^\rho}\, w^*, \quad d(1) \neq 0, \tag{3.9}$$

where ρ is the degree of the polynomial $d(q) - r(q)$. These inequalities already do not include the controller parameters and are defined only by the plant properties.

3.3. Harmonic Perturbations

We assume that the external perturbation is a harmonic function like $w(k) = w^* \sin(\omega kT)$, where $w^* > 0$ is the perturbation amplitude; ω is the circular (generally speaking, unknown) frequency; and T is the discreteness (quantization) period of controller (3.2). Then, according to the classical theory of linear discrete-time systems [12, 17, 24, 25], the amplitude $a > 0$ of the output steady-state oscillations of the controlled variable

$$\lim_{k\to\infty} y(k) = a \sin(\omega kT + \phi), \tag{3.10}$$

where ϕ is the phase shift relative to the input harmonic, can be established from

$$a = |T_{yw}(e^{j\omega T})|\ w^* = \frac{|w_f(e^{j\omega T})|}{|1 + w_p(e^{j\omega T})|}\ w^*, \tag{3.11}$$

where $T_{yw}(q)$ is the transfer function of the closed-loop system (3.1), (3.2) from (3.4).

The denominator of (3.11) comprises the magnitude of the recurrent difference calculated at the frequency of external perturbation. The greater the magnitude, the smaller the amplitude a of the forced oscillations of the controlled variable y. In the discrete-time case, however, this value is always bounded from above, and, consequently, there exists the maximum achievable amplitude of oscillations of the controlled variable which in essence cannot be reduced with the use of the output controller (3.2).

Theorem 5. *Under the harmonic external perturbation* $w(k) = w^* \sin(\omega kT)$*, for all real frequencies* $\omega \in [0, \pi/T]$ *the recurrent difference of system* (3.1), (3.2) *with the discrete-time output controller and the maximum achievable control error amplitude satisfy the inequalities*

$$|1 + w_p(e^{j\omega T})| < \frac{2^n}{|d_p(e^{j\omega T})|}, \quad d_p(e^{j\omega T}) \neq 0, \tag{3.12}$$

$$a > \frac{|w_f(e^{j\omega T})|\ |d_p(e^{j\omega T})|}{2^n}\ w^*. \tag{3.13}$$

Theorem 5 is proved in the Appendix.

We note that the inequalities (3.7) and (3.8) follow from Theorem 5 under the zero frequency of the external perturbation $\omega = 0$ and emphasize that the inequalities of this theorem are valid independently of the minimal phase property of the plant. At the same time, if the plant is a minimal phase one, then as the result of cancellation by the polynomial $b(q)$ in (3.4) and (3.6) for $g(q) = b(q)$ and $D(q) = b(q)[d(q) - r(q)]$ these inequalities acquire the form

$$|1 + w_p(e^{j\omega T})| < \frac{2^\rho}{|d(e^{j\omega T})|}, \quad a > \frac{|l(e^{j\omega T})|}{2^\rho}\ w^*, \quad d(e^{j\omega T}) \neq 0, \tag{3.14}$$

where ρ is the degree of the polynomial $d(q) - r(q)$. The right sides of these inequalities do not involve the controller parameters and are defined only by the properties of the plant polynomials.

4. CONCLUSIONS

It was demonstrated that, in contrast to the continuous case, in the systems with discrete-time controllers there always exists a finite limit to the achievable precision of control. This is dues to the fact that the transfer coefficient in the state of open loop for the system with a single control, as well as the singular values of the matrix of recurrent difference which in the multidimensional case define the steady-state control errors are in essence bounded values by virtue of the specificity of the modal stability domain of the discrete-time systems. It is obvious that this basic constraint becomes more prominent with reduction of the system order and increase in the quantization period of the discrete-time controller. These results make it clear why it is impossible to provide an infinite stability margin in the gain in the procedures of LQ-optimization of the discrete-time systems [4, 7, 8, 30, 31].

The standard bounded vector time functions such as the step-type and harmonic functions of unknown frequency were considered as the external perturbations. The only requirement on the controllers, besides their physical realizability, that was used at deriving the basic results was the condition for asymptotic stability of the closed-loop system. Therefore, the conclusions of the present paper apply to the entire set of the discrete-time stabilizing controllers, no matter what methods were used to design them. The results obtained are as follows.

1. It was proved that in the systems with discrete-time controllers there always exists a finite limit for the achievable control precision.
2. For the systems with single control actions, established were the achievable estimates of the magnitude of the frequency transfer function of the open-loop system (gain under the step-type perturbations) and the level of minimal possible control errors that are independent of the parameters of the discrete-time controller and defined by the system order, value of the quantization period, and the plant properties.
3. For the multidimensional systems, a similar achievable estimate of the singular values of the frequency transfer matrix of the recurrent difference defining the limit control errors of systems with more than one control actions was obtained.

These results are readily generalized to a wider class of polyharmonic perturbations with unknown amplitudes and power-limited frequencies.

APPENDIX

Proof of Theorem 1. For $m = m_1 = 1$ and $w = 0$, we represent the equations of the closed-loop system as

$$\begin{bmatrix} qI_n - A & -B_2 \\ -K & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = 0.$$

Then, with regard for the well-known rule of expansion of the determinant of the block matrix [32, 33], the characteristic polynomial $D(q)$ of the closed-loop system (2.1), (2.3) is given by

$$D(q) = \det \begin{bmatrix} qI_n - A & -B_2 \\ -K & 1 \end{bmatrix} = \det(qI_n - A)[1 - K(qI_n - A)^{-1}B_2],$$

where $\det(qI_n - A) = d(q)$ is the characteristic polynomial of the open-loop system coinciding here with characteristic polynomial of the plant. Whence follows the identity

$$1 + w_p(q) = \frac{D(q)}{d(q)}. \tag{A.1}$$

By expressing the characteristic polynomial of the closed-loop system

$$D(q) = (q - q_1)(q - q_2)\ \dots(q - q_n)$$

in terms of its roots q_i ($i = \overline{1, n}$), with regard for (2.8) we establish from the last identity that for $q = 1$

$$|1 + k_p| = \frac{|(1 - q_1)(1 - q_2)\ \ldots (1 - q_n)|}{|d(1)|}. \tag{A.2}$$

Since the closed-loop system (2.1), (2.3) is asymptotically stable

$$|q_i| < 1, \quad i = \overline{1, n}, \tag{A.3}$$

the inequality $|(1 - q_i)| < 2$ is satisfied for each real root q_i of the polynomial $D(q)$. If the root $q_i = \operatorname{Re} q_i + j \operatorname{Im} q_i$ of the polynomial $D(q)$ is complex, then it appears in the numerator of (A.2) together with its complex conjugate value $\overline{q}_i$. On this ground we establish

$$\begin{gathered} (1 - q_i)(1 - \overline{q}_i) = (1 - \operatorname{Re} q_i - j \operatorname{Im} q_i)(1 - \operatorname{Re} q_i + j \operatorname{Im} q_i) \\ = (1 - \operatorname{Re} q_i)^2 + \operatorname{Im}^2 q_i = 1 + \operatorname{Re}^2 q_i + \operatorname{Im}^2 q_i - 2\operatorname{Re} q_i < 2 - 2\operatorname{Re} q_i \\ = 2(1 - \operatorname{Re} q_i) < 2^2. \end{gathered}$$

The signs of inequality are put here in virtue of (A.3). On the basis of the aforesaid, the following inequality is true for the numerator in the right side of (A.2):

$$|D(1)| = |(1 - q_1)(1 - q_2)\ \ldots (1 - q_n)| < 2^n \tag{A.4}$$

from which inequality (2.9) follows. The second inequality (2.10) follows from (2.7) with regard for (2.9), which proves Theorem 1.

Proof of Theorem 2. Having first expressed the polynomial $D(q)$ in terms of its roots

$$D(e^{j\omega T}) = (e^{j\omega T} - q_1)(e^{j\omega T} - q_2)\ \ldots (e^{j\omega T} - q_n),$$

we estimate the numerator in the right side of identity (A.1) for $q = e^{j\omega T}$.

In virtue of condition (A.3), for asymptotic stability of the closed-loop system each of the multipliers of the last expression satisfies the inequality

$$|e^{j\omega T} - q_i| < 2, \quad i = \overline{1, n}.$$

One can readily make sure of this fact from the geometrical interpretation of the expression in the left side of the last inequality which represents the length of the radius vector drawn from the point q_i belonging to the interior of the unit circle to the point $e^{j\omega T}$ lying on the unit circle centered at the origin (see the figure).

Taking the last inequality into account, we get the estimate

$$|D(e^{j\omega T})| < 2^n \tag{A.5}$$

enabling us to establish from identity (A.1) the first of the proved inequalities. The second inequality follows with regard for (2.16) from (2.15) for the amplitude of the steady-state oscillations of the controlled variable, which proves Theorem 2.

Proof of Theorem 3. The characteristic polynomial $D(q)$ of the closed-loop system (2.1), (2.3) is representable as in [7]:

$$D(q) = \det \begin{bmatrix} qI_n - A & -B_2 \\ -K & I_m \end{bmatrix} = \det(qI_n - A) \det[I_m - K(qI_n - A)^{-1}B_2].$$

This expression was obtained using the rules for expansion of the determinant of the block matrix from [34]. Hence we get the identity

$$\det[I_m + W_p(q)] = \frac{D(q)}{d(q)}, \quad d(q) = \det(qI_n - A) \tag{A.6}$$

represent its left side as [33]:

$$\det[I_m + W_p(q)] = \prod_{i=1}^{m} \lambda_i[I_m + W_p(q)],$$

where $\lambda_i[V]$ is the ith eigenvalue of the matrix V, and make use of the equality from [33]:

$$\prod_{i=1}^{m} |\lambda_i(M)| = \prod_{i=1}^{m} \sigma_i(M),$$

where $\sigma_i(M)$ is the ith singular value of the matrix M.

Assuming that in identity (A.6) $q = 1$ and relying on the last equalities and previously proved inequality (A.4) which is valid in virtue of the asymptotic stability of the closed-loop system (2.1), (2.3), we put down the following string of equalities and inequalities

$$\begin{gathered}
|\det[I_m + W_p(1)]| = \prod_{i=1}^{m} |\lambda_i[I_m + W_p(1)]| = \prod_{i=1}^{m} \sigma_i[I_m + W_p(1)]; \\
|\det V(1)| = \prod_{i=1}^{m} \sigma_i[I_m + W_p(1)] = \frac{|D(1)|}{|d(1)|} < \frac{2^n}{|d(1)|}; \\
(\sigma_{\min}[I_m + W_p(1)])^m \leqslant \prod_{i=1}^{m} \sigma_i[I_m + W_p(1)] < \frac{2^n}{|d(1)|}
\end{gathered} \tag{A.7}$$

from the last of which (2.22) follows.

Passing now to the proof of inequality (2.23), we represent the equation of hyperellipsoid (2.19) in the equivalent form $z^{\mathrm{T}}(\infty)Mz(\infty) = 1$, where M is the symmetrical positive definite $(m \times m)$ matrix:

$$M = [W_0{}^{-1}(1)]^{\mathrm{T}}\, [I_m + W_p(1)]^{\mathrm{T}}\, [I_m + W_p(1)]\, W_0{}^{-1}(1)/\|w^*\|^2.$$

The volume of ellipsoid obeys the following formula [27]:

$$V_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)} (\det M)^{-1/2}, \tag{A.8}$$

where the numerical coefficient including the gamma function is defined for the even and odd values of m, respectively by

$$\frac{\pi^p}{p!}, \quad m = 2p; \quad 2\frac{(2\pi)^p}{(2p+1)!!}, \quad m = 2p+1.$$

Taking into consideration the fact that the matrix M is a product of multipliers and the well-known properties of the matrix determinants [27], we establish from (A.8) that

$$V_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)} \frac{|\det W_0(1)|}{|\det V(1)|} \|w^*\|^m.$$

With regard for (A.7), the second of the considered inequalities follows from the last expression.

Inequalities (2.22) and (A.7), obviously, make sense if $d(1) \neq 0$. Otherwise, one can resort to the method similar to that of Remark 1. This proves Theorem 3.

Proof of Theorem 4. Using the relation between the determinant of the matrix with the product of eigen and singular values [27, 33] (see the proof of Theorem 3), we establish for $q = e^{j\omega T}$ an analog of the first equality of (A.7):

$$|\det[I_m + W_p(e^{j\omega T})]| = \prod_{i=1}^{m} |\lambda_i[I_m + W_p(e^{j\omega T})]| = \prod_{i=1}^{m} \sigma_i[I_m + W_p(e^{j\omega T})]. \tag{A.9}$$

Taking this equality into account and assuming that $q = e^{j\omega T}$ in identity (A.6) and using also the property of asymptotic stability of the closed-loop system (2.1), (2.3) in virtue of which inequality (A.5) is valid, we obtain

$$|\det V(e^{j\omega T})| = \prod_{i=1}^{m} \sigma_i[I_m + W_p(e^{j\omega T})] = \frac{|D(e^{j\omega T})|}{|d(e^{j\omega T})|} < \frac{2^n}{|d(e^{j\omega T})|} \tag{A.10}$$

from which we conclude that the inequality

$$(\sigma_{\min}[I_m + W_p(e^{j\omega T})])^m \leqslant \prod_{i=1}^{m} \sigma_i[I_m + W_p(e^{j\omega T})] < \frac{2^n}{|d(e^{j\omega T})|},$$

takes place which gives rise to (2.33).

Passing to the proof of inequality (2.34), we notice that the volume of hyperellipsoid (2.32) obeys (A.8) where $2m$ must be substituted for m. Then, with regard for the equalities $\Gamma(m+1) = m!$ and $\det M = |\det Q|^2$ and the structure of the matrix Q, we obtain from (2.31)

$$V_{2m} = \frac{\pi^m}{m!} \frac{|\det W_0(e^{j\omega T})|^2}{|\det V(e^{j\omega T})|^2} \|w^*\|^{2m}.$$

Taking into consideration inequality (A.10), we obtain from this expression the second of the considered inequalities, which proves Theorem 4.

It follows from the above relations (A.9), (A.10) that there are bounded numbers in the denominators of inequalities (2.30). Therefore, the lower boundary of the norm of the amplitude vector of control errors in the multidimensional discrete-time systems with state controllers is always finite.

Proof of Theorem 5. In virtue of the previously proved inequality (A.5), the first inequality of Theorem 5 follows from identity (3.6) because the polynomial $D(q)$ is stable. With regard for this inequality, the second inequality follows from (3.11) for the amplitude of the steady-state oscillations of the controlled variable, which proves Theorem 5.

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