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H_∞-Approach to Controller Synthesis under Parametric Uncertainty and Polyharmonic External Disturbances

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Abstract—We consider the robust stabilization problem for linear multivariable systems whose physical parameters may deviate from computed (nominal) in some known bounds, and the control object is subject to non-measurable polyharmonic external disturbances (with unknown amplitudes and frequencies) bounded in power. We pose the problem of synthesizing a controller that guarantees robust stability of the closed-loop system and additionally ensures given errors with respect to controlled variables in the established nominal mode. The solution of this problem is based on the technique of opening the object–controller system with respect to varying object parameters and can be reduced to a standard H_∞-optimization procedure, while the necessary accuracy is achieved by choosing the weight matrix for controlled object variables. We show the solution for a well-known benchmark problem.

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1. INTRODUCTION

The robust stabilization problem for systems with parameters deviating from computed (nominal) values has been the subject of a large number of studies; see, e.g., [1–5]. At the same time, most works usually consider as model parameters either elements of state equation matrices or coefficients of transfer functions that comprise the object's transfer matrix. Generally speaking, these parameters are not *physical* since state equations and the transfer matrix are secondary forms for describing dynamical systems. They result from a transformation of the original equations of the dynamical system based on fundamental physical laws of mechanics and electrodynamics. In this work, we consider parameters of this original description form that have a clear physical meaning (mass, moment of inertia, resistance, capacity, inductivity and so on). Besides, a transition from the original description form in physical variables to a different form is usually accompanied by "mixing" and "multiplying" the varied parameters which significantly complicates the original problem and makes the end result much more conservative [1].

Usually real life dynamical systems are subject to non-measurable external disturbances which in mathematical control theory are bounded with respect to some norm [4–7]. Under disturbances, controlled variables of the control object deviate from their nominal values (which are zero in the stabilization problem), and therefore there arises the problem of providing given deviations (no more than is admissible) of these variables from zero [6]. The problem of suppressing external disturbances has been the subject of many studies [4–9]. In particular, the work [7] develops the method of invariant ellipsoids for external disturbances bounded either in Euclidean norm or componentwise at every time moment. One also has to take into account constraints on controlling signals and nonzero initial conditions. Theoretical and numerical results are based on the method of linear matrix inequalities (LMI) [5].

Similar directions of study, initiated by publications [8, 9], present a certain additional circuit that lets one estimate the external disturbance and then compensate for it. Here we make various assumptions regarding the object's properties: stable [8] and/or minimal phase [8, 9]. One typical complication in these approaches is that a part of controlling device parameters can be found during its synthesis only at the stage of mathematical modeling for the closed-loop system.

The approach developed in this work is based on representing the dynamical system in the so-called canonical $(W,\ \Lambda,\ K)$-form [1, 10, 11] (first proposed in [10]), where physical parameters subject to deviations from the computed ones form internal dummy feedback in the form of a diagonal matrix Λ. This method, thanks to its engineering clarity and simple numerical implementation, can be viewed as a significantly simpler alternative to J. Doyle's μ-approach [4, 12–15], which is very computationally intensive, not too efficient for real-valued uncertainties (real-μ) [4, 13], and leads to controllers of very high order (see, e.g., [15], where a second order object has a controller of order 22!).

As non-measurable external disturbances we consider polyharmonic power-bounded disturbances. Similar to [16, 17], we use the notion of a radius of the steady state for a dynamical system with respect to controlled variables (which characterizes accuracy), and the controller, apart from robust stability, must also ensure a given radius or minimize it.

Our solution for the robust stabilization problem is original; it is based on a frequency matrix inequality that determines multivariable gain margins after the closed system breaking with respect to parameters (with a dummy control variable $\widetilde{u}$ in the $(W,\ \Lambda,\ K)$-form), unlike classical opening points: physical object input or output. This frequency condition reduces to standard problem of meeting external disturbances in the H_∞ approach [1], and the given accuracy is achieved with a special choice of the weight matrix for controlled object variables similar to [16, 17]. We have implemented this approach in the MATLAB software suite with the Robust Control Toolbox [12]; our implementation uses the LMI technique. We show a sample controller synthesis for a wide known benchmark problem from [1, 12, 13].

2. STATEMENT OF THE PROBLEM

Consider a control object defined by the following equations in physical variables:

$$\begin{gathered} L_1(p)z_f(t) = L_2(p)u(t) + L_3(p)f(t), \\ y(t) = Nz_f(t), \end{gathered} \tag{2.1}$$

where z_f is the l-dimensional vector of physical object variables (velocity, acceleration, current, voltage, movement, angle of rotation and so on); u, an m-dimensional vector of control influences; y, an m_2-dimensional vector of measured (and at the same time controlled) object variables; f, a μ-dimensional vector of external bounded non-measurable disturbances; N, a known numerical matrix of size $m_2 \times l$; $L_1(p), L_2(p), L_3(p)$, polynomial matrices of size $l \times l$, $l \times m$, and $l \times \mu$ respectively of the differentiation operator $p = d/dt$:

$$L_1(p) = \sum_{i=0}^{\alpha_1} L_1^{(i)} p^i, \quad L_2(p) = \sum_{j=0}^{\alpha_2} L_2^{(j)} p^j, \quad L_3(p) = \sum_{k=0}^{\alpha_3} L_3^{(k)} p^k, \tag{2.2}$$

where $L_1^{(i)},\ L_2^{(j)},\ L_3^{(k)}$ are known real matrices of the corresponding dimensions, $\alpha_2,\ \alpha_3 < \alpha_1$.

We will assume that object (2.1) is stabilizable and detectable. In what follows we call elements of matrices $L_1^{(i)}$ $(i = \overline{1,\alpha_1})$, $L_2^{(j)}$ $(j = \overline{1,\alpha_2})$ physical parameters of the control object. Suppose that n physical parameters of the object (their number and location in the matrices is not constrained) with nominal values $\lambda_1, \lambda_2, \ldots, \lambda_n$ can take values from given intervals

$$\lambda_i + \Delta\lambda_i \in (\lambda_i^{\min}, \quad \lambda_i^{\max}), \quad i = \overline{1,n}, \tag{2.3}$$

where $\Delta\lambda_i$ is the deviation of a parameter from nominal values, $\lambda_i^{\min}$, $\lambda_i^{\max}$ are known lower and upper bounds.

Elements of matrix $L_3(p)$ do not influence the stability of the closed-loop system, so their deviation from nominal values are not considered below.

Components of the external disturbances vector f are bounded polyharmonic functions

$$f(t) = \sum_{k=1}^{p_0} w_{ik} \sin(\omega_k t + \psi_{ik}), \quad i = \overline{1, \mu}. \tag{2.4}$$

Here amplitudes w_{ik}, initial phases ψ_{ik} ($i = \overline{1, \mu}$, $k = \overline{1, p_0}$), and frequencies ω_k ($k = \overline{1, p_0}$) of the harmonics are not known, but it is known that amplitudes of the harmonics are subject to the following condition (that bounds the power of each external disturbance component):

$$\sum_{k=1}^{p_0} w_{ik}^2 \leqslant (w_i^*)^2, \quad i = \overline{1, \mu}, \tag{2.5}$$

where p_0 is a known number of harmonics, w_i^* ($i = \overline{1, \mu}$) are given numbers.

We define steady state errors with respect to controlled variables with a relation from [6, 16]: $y_{i,st} = \lim_{t\to\infty} \sup|y_i(t)|$ ($i = \overline{1, m_2}$). Usually we want to find such stabilizing output controller that inequalities (requirements on accuracy) $y_{i,st} \leqslant y_i^*$ ($i = \overline{1, m_2}$) hold, where $y_i^* > 0$, $i = \overline{1, m_2}$, are given numbers. However, it is clear that such a controller may not exist, so we define the radius of the steady state of the closed-loop system with respect to controlled variables with the following relation:

$$r_{st}^2 = \sum_{i=1}^{m_2} \left(\frac{y_{i,st}}{y_i^*} \right)^2, \tag{2.6}$$

and we will bound this radius [16, 17].

Problem 1. Construct a stabilizing output controller

$$u(t) = K(p) y(t) \tag{2.7}$$

such that, on one hand, for given finite deviations of parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ from computed (2.3) the closed-loop system (2.1), (2.7) preserves asymptotic stability, and on the other hand it holds that

$$r_{st}^2 = \sum_{i=1}^{m_2} \left(\frac{y_{i,st}}{y_i^*} \right)^2 \leqslant \gamma^2, \tag{2.8}$$

where $K(p)$ is the controller's transfer matrix whose elements are regular rational functions of the operator p; γ is a given or minimized number.

It is easy to see that the equality sign in expression (2.8) (for given γ) corresponds to the equation of a hyperellipsoid with given semiaxes whose surfaces belong to control errors. If as a result of solving the synthesis problem we get $\gamma \leqslant 1$, then, obviously, requirements on the accuracy are satisfied as well.

3. REDUCING THE SYSTEM TO CANONICAL (W, Λ, K)-FORM

To solve the problem posed above, we represent equations of the closed-loop system (2.1), (2.7) in canonical (W, Λ, K)-form [1] accounting for the external disturbance f:

$$\begin{aligned} \widetilde{y} &= W_{11}\widetilde{u} + W_{12}u + W_{13}f, \quad \widetilde{u} = \Lambda\widetilde{y}, \\ y &= W_{21}\widetilde{u} + W_{22}u + W_{23}f, \quad u = K(s)y, \end{aligned} \tag{3.1}$$

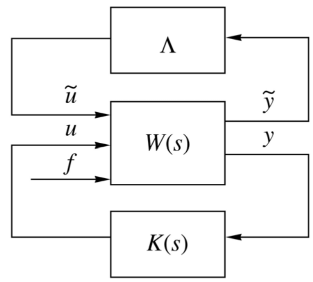

Fig. 1.

where $W_{ij}(s)$ $(i = 1, 2,\ j = \overline{1,3})$ are known transfer matrices that do not contain varied parameters (2.3); u, y are the physical input and output of the control object (2.1); $\widetilde{u}, \widetilde{y}$ are n-dimensional dummy input and output of the control object; $\Lambda = \mathrm{diag}[\lambda_1,\ \lambda_2, \ldots, \lambda_n]$ is the diagonal matrix of control object parameters subject to deviations from nominal values; $K(s)$ is the controller's transfer matrix (2.7) that we are looking for, where s denotes the Laplace transform.

The block-diagram of the $(W,\ \Lambda,\ K)$-form corresponding to Eqs. (3.1) is shown on Fig. 1.

Theorem 1. *Equations of the closed-loop system* (2.1), (2.7) *can always be represented in equivalent* (W, Λ, K)*-form* (3.1).

Proof of Theorem 1 is constructive and yields an algorithm for constructing the (W, Λ, K)-form; it is given in the Appendix.

4. AN APPROACH TO SOLVING THE PROBLEM [1]

The transfer matrix of the open-loop system (3.1) with respect to varied parameters λ_i $(i = \overline{1,n})$ (when we open the system with respect to variable $\widetilde{u}$) is written in the following form from [1]:

$$W^{\widetilde{u}}_{\text{open}}(s) = \Lambda \left[-W_{11} - W_{12}K(I - W_{22}K)^{-1}W_{21}\right], \tag{4.1}$$

where I is the unit matrix of the corresponding dimensions.

As we can see in (4.1), an important characteristic feature of this transfer matrix is that the varied parameters form in it a diagonal matrix of gains.

If this transfer matrix satisfies a circular frequency inequality [10, 11]

$$\left[I + W^{\widetilde{u}}_{\text{open}}(-j\omega)\right]^{\mathrm{T}} \left[I + W^{\widetilde{u}}_{\text{open}}(j\omega)\right] \geqslant r^2 I, \quad \omega \in [0, \infty) \tag{4.2}$$

(where r is the stability margin radius $0 < r \leqslant 1$), then the following sufficient estimates on the interval of possible values of parameters hold [1, 10, 11]:

$$\min\left\{\frac{\lambda_i}{1+r}, \frac{\lambda_i}{1-r}\right\} < \lambda_i + \Delta\lambda_i < \max\left\{\frac{\lambda_i}{1+r}, \frac{\lambda_i}{1-r}\right\}, \quad i = \overline{1,n} \tag{4.3}$$

and guarantee robust stability of the closed-loop system (2.1), (2.7).

In case $n = 1$ inequality (4.2) means [1] that the Nyquist diagram $W^{\widetilde{u}}_{\text{open}}(j\omega)$ does not intersect a circle of radius r centered at the critical point $(-1, j0)$ on the hodograph plane. In case $n > 1$ this frequency condition has the following physical interpretation: with respect to each dummy object input $\widetilde{u}_i (i = \overline{1,n})$ (see Fig. 1) gains can be changed from the nominal value equal to one independently of each other in intervals $(1/(1+r), 1/(1-r))$ without any loss of stability (similar to the case $n = 1$). This implies the bounds (4.3) due to the diagonal structure of matrix Λ [1, 10, 11], since a change of the gains in the circuit can be recalculated into a deviation of the parameter itself from its nominal value.

Thus, solving the first part of Problem 1 reduces to such a construction of matrix K for controller (2.7) for which r in (4.2) takes a given value or is maximized. If under chosen nominal values λ_i ($i = \overline{1, n}$) given intervals for parameters (2.3) turn out to be included into the corresponding intervals (4.3), it means that the first part of Problem 1 is already solved. This problem has been considered in [1]. The next step in solving Problem 1 is to additionally take into account condition (2.8).

5. REDUCING THE PROBLEM TO A STANDARD H_∞-OPTIMIZATION PROBLEM

Consider a closed-loop system presented on Fig. 2 and defined by the following equations:

$$\begin{aligned} &\widetilde{y} = W_{11}z_1 + W_{12}u + W_{13}f, \quad \widetilde{u} = \Lambda\widetilde{y}, \quad z_1 = \widetilde{u} + w_1, \\ &y = W_{21}z_1 + W_{22}u + W_{23}f, \quad u = Ky, \quad z_2 = Q^{1/2}y, \end{aligned} \tag{5.1}$$

where $w_1 \in R^n$ is the vector of dummy external disturbances; $z_1 \in R^n$ is the vector of dummy controlled variables; $z_2 \in R^{m_2}$ is the weighted vector of variables controlled with the diagonal matrix $Q = \mathrm{diag}[q_1, q_2, \ldots, q_{m2}]$ with positive elements $q_i > 0$ ($i = \overline{1, m_2}$). Compared to the work [1], here we additionally have an external disturbance f and a controlled variable z_2.

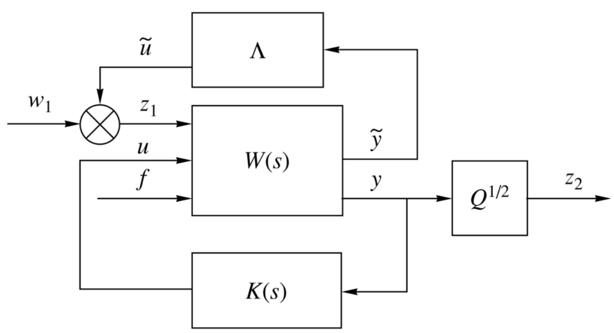

Fig. 2.

We introduce the extended vector of external disturbances w which includes the vector of dummy external disturbances w_1 and the vector of disturbances f, together with the extended vector of controlled variables z that unites vectors z_1 and z_2. We denote the transfer matrix of the closed-loop system that relates these vectors by T_{zw}. Then we can write

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T_{zw}w = \begin{bmatrix} T_{z_1w_1} & T_{z_1f} \\ Q^{1/2}T_{yw_1} & Q^{1/2}T_{yf_1} \end{bmatrix} \begin{bmatrix} w_1 \\ f \end{bmatrix}, \tag{5.2}$$

where $T_{z_1w_1}$ is the transfer matrix that relates vectors w_1 and z_1 in the closed-loop system (5.2), and the other elements T_{zw} are defined similarly.

Suppose that the stabilizing controller in question $K(s)$ minimizes the H_∞-norm of the transfer matrix for the closed-loop system (5.2):

$$\|T_{zw}\|_\infty \leqslant \gamma. \tag{5.3}$$

Then each block in this matrix satisfies a similar condition [14], in particular,

$$\|T_{z_1w_1}\|_\infty \leqslant \gamma \quad \text{and} \quad \|Q^{1/2}T_{yf}\|_\infty \leqslant \gamma. \tag{5.4}$$

The first inequality from (5.4) can be represented as condition (4.2), where according to [1] we have

$$T_{z_1 w_1}(s) = [I + W_{\text{open}}^{\widetilde{u}}(s)]^{-1}, \quad \text{and} \quad r = 1/\gamma. \tag{5.5}$$

The second inequality from (5.4) can also be represented in equivalent frequency form [6, 17]:

$$T_{yf}^{\mathrm{T}}(-j\omega) Q T_{yf}(-j\omega) \leqslant \gamma^2 I, \quad \omega \in [0, \infty). \tag{5.6}$$

The frequency inequality (5.6) together with the diagonal structure of matrix Q and the lemma on steady-state values [6, 17] yields an inequality for the steady-state values of controlled variables

$$\sum_{i=1}^{m_2} q_i y_{i,st}^2 \leqslant p_0 \gamma^2 \|w^*\|^2, \tag{5.7}$$

where $\|w^*\|$ is the Euclidean norm of vector w^* with components of the right-hand side of (2.5).

Choosing the elements of the diagonal weight matrix Q from equalities

$$q_i = \frac{p_0 \|w^*\|^2}{(y_i^*)^2}, \quad i = \overline{1, m_2}, \tag{5.8}$$

we arrive at the objective condition (2.8), where γ is the realized value for the numerical solution of problem (5.3).

We can reformulate the problem of finding a correct transfer matrix for controller $K(s)$ that would satisfy inequalities (2.8) and (4.2) in the form of the following auxiliary problems of H_∞-optimal and suboptimal control.

Problem 2. Find a correct transfer matrix of controller (2.7) that would ensure inequality (5.3) with minimal possible $\gamma = \gamma_0$.

Problem 3. Given a number $\gamma > \gamma_0$, find a correct transfer matrix of controller (2.7) such that inequality (5.3) holds.

If Problems 2 or 3 have been solved, we can find from (4.3) sufficient estimates on the interval of possible parameter values that guarantee robust stability of system (2.1), (2.7), where the stability margin radius $r = \gamma^{-1}$, and inequality (2.8) implies estimates on control errors that would certainly be no worse than

$$y_{i,st} \leqslant \gamma\, y_i^*, \quad i = \overline{1, m_2}. \tag{5.9}$$

Comparing the resulting intervals with given intervals, we can conclude that Problem 1 has been successfully solved. Thus, we can formulate the following statement.

Theorem 2. *Suppose that controller* (2.7) *resolves the auxiliary H_∞-problem* (5.3) *when diagonal elements of the weight matrix Q are taken from equalities* (5.8). *Then the value of γ realized in the numerical solution of this problem will determine: a) the stability margin radius $r = \gamma^{-1}$ that guarantees sufficient robust stability boundaries* (4.3); *b) sufficient estimates on control errors* (2.8), *and* (5.9).

Let us now write Eqs. (5.1) in a standard form used in H_∞-control theory:

$$z = G_{11} w + G_{12} u, \quad y = G_{21} w + G_{22} u, \quad u = K y,$$

where $G_{ij}(s)$ $(i, j = 1, 2)$ are blocks of the generalized object's transfer matrix $G(s)$.

Assertion. *Transfer matrices* $G_{ij}(s)$ $(i,j=1,2)$ *of the generalized object are related to transfer matrices of Eqs.* (5.1) *by the following equations:*

$$G_{11}=\begin{bmatrix}(I-\Lambda W_{11})^{-1} & (I-\Lambda W_{11})^{-1}\Lambda W_{13}\\ Q^{1/2}W_{21}(I-\Lambda W_{11})^{-1} & Q^{1/2}W_{21}(I-\Lambda W_{11})^{-1}\Lambda W_{13}+Q^{1/2}W_{23}\end{bmatrix},$$
$$G_{12}=\begin{bmatrix}(I-\Lambda W_{11})^{-1}\Lambda W_{12}\\ Q^{1/2}W_{21}(I-\Lambda W_{11})^{-1}\Lambda W_{12}+Q^{1/2}W_{22}\end{bmatrix},$$
$$G_{21}=\begin{bmatrix}W_{21}(I-\Lambda W_{11})^{-1} & W_{21}(I-\Lambda W_{11})^{-1}\ \Lambda W_{13}+W_{23}\end{bmatrix},$$
$$G_{22}=\begin{bmatrix}W_{21}(I-\Lambda W_{11})^{-1}\Lambda W_{12}+W_{22}\end{bmatrix}. \tag{5.10}$$

The proof of this assertion is elementary and is not shown here.

Thus, solving the H_∞-problem (5.3) we solve the original Problem 1.

Note that from the computational point of view it makes sense for the solution of problem (5.3) to reduce generalized object to state equations [1]:

$$\begin{aligned}\dot{x}&=Ax+B_1w+B_2u,\\ z&=C_1x+D_{11}w+D_{12}u,\\ y&=C_2x+D_{21}w+D_{22}u,\end{aligned} \tag{5.11}$$

where dimension of the state vector x coincides with the degree of characteristic of the polynomial object (2.1): $\det L_1(s)$. It is obvious now that the vector of controlled variables z will not contain control influences, and there are no measurement noises. This means that problem (5.3) is singular [1, 18] and, consequently, cannot be solve with the 2-Riccati approach [19]. To solve such a singular problem numerically, it is convenient to use the method of linear matrix inequalities implemented in a MATLAB package [12] (similar to [1, 20]).

6. THE SYNTHESIS PROCEDURE

We represent the synthesis procedure as a sequence of actions.

1. Reduce equations of system (2.1), (2.7) to the form (3.1), where Λ is the diagonal matrix that includes nominal physical parameters of the system chosen by the designer, parameters that will be subject to deviations from nominal values.

2. Write Eqs. (5.1) in standard form which is common in H_∞-control theory taking into account (5.10) and reduce this form to state Eqs. (5.11).

3. Solve H_∞-control Problems 2 and 3 (5.3) together with (5.8) and (5.11) and find the transfer matrix $K(s)$ of controller (2.7).

4. Find the boundaries of guaranteed margins on the object parameters with formulas (4.3) and find estimates on control errors from (2.8) or (5.9).

5. Compare the margin boundaries found on step 4 with given ones.

6. If the resulting margin boundaries on object parameters from (4.3) do not cover the ones given in (2.3) or the resulting bounds on control errors exceed given ones, this method does not yield a solution. However, one can choose other values of nominal parameters or decrease the desired values of control errors and repeat the synthesis starting from step 2. But in the general case, for instance, if the object is not minimal phase with respect to control and given errors are less than minimal possible (or intervals (2.3) are so wide that no linear controller can stabilize the system),

the solution is not guaranteed to succeed. Numerical experiments have shown that in the case of such "failure" of the method one or two iterations usually suffice to see that control objectives are impossible to achieve.

Let us comment upon individual steps of this algorithm. In the software package [12], controller corresponds to a quadruple of matrices (A_c, B_c, C_c, D_c) that define its state equations, and the controller's transfer matrix that we are looking for has the form $K(s) = C_c(sI - A_c)^{-1}B_c + D_c$. The order of the controller does not exceed the order of the object, i.e., the degree of polynomial $\det L_1(s)$. During step 3 of the procedure, we first find the minimal possible value $\gamma = \gamma_0$ in problem (5.3) with the *hinflmi* function (see [17] for details) and then solve the suboptimal problem (also with *hinflmi*) for $\gamma > \gamma_0$ (for $\gamma = \gamma_0$ a part of eigenvalues of the matrix of the closed-loop system (2.1), (2.7) turns out to be nearly at the imaginary axis!) and find $K(s)$ itself. During step 4, to find control errors one can use mathematical modeling, as we will show in the example below.

7. A SAMPLE SYNTHESIS PROBLEM'S SOLUTION

We illustrate the proposed approach to synthesis with the example of a two-mass system with elastic connection: two wagons connected by a spring. The model is defined by the following equations [13]:

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -qx_1 + qx_2 + u + f, \quad \dot{x}_4 = qx_1 - qx_2 y = x_2, \tag{7.1}$$

where q is the varied parameter (spring rigidity) whose nominal value equals 0.8; x_1 is the movement of the first wagon; x_2 is the movement of the second wagon; $y = x_2$ is the measured variable; u is the control influence; f is the external disturbance.

Reduction of Eqs. (7.1) and (2.7) to form (5.1) repeats the derivation shown in [1] (where $\Lambda = q$ is scalar), only instead of u we use the sum $u + f$, so we do not show it here. A similar remark can be also made regarding formulas (5.10). Using state equations shown in [1], it is easy to get state Eqs. (5.11) that correspond to (5.1), where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q & q & 0 & 0 \\ q & -q & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -q \;\; q \;\; 0 \;\; 0 \\ Q^{1/2} \times (\, 0 \;\; 1 \;\; 0 \;\; 0 \,) \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{22} = [0].$$

Suppose that the steady state error in the controlled variable must not exceed $y^* = 1$, and the external disturbance (2.4), (2.5) is bounded by $w^* = 1$. Then according to (5.8) we get the value of the only weight coefficient at the scalar controlled variable y on the block-diagram shown on Fig. 2, $Q = q_1 = 1$.

The controller $K(s)$ obtained with software package [12] has the form

$$\frac{-3.1 \times 10^8 s^3 - 4.2 \times 10^8 s^2 - 4.6 \times 10^8 s - 1.6 \times 10^8}{s^4 + 193.7 s^3 + 2.2 \times 10^4 s^2 + 1.4 \times 10^6 s + 6.1 \times 10^7}.$$

Note that amplitude–frequency characteristics of the closed-loop system $|T_{yf}(j\omega)|$ shown on Fig. 3 (with a controller constructed with the method proposed here) and on Fig. 4 (obtained with

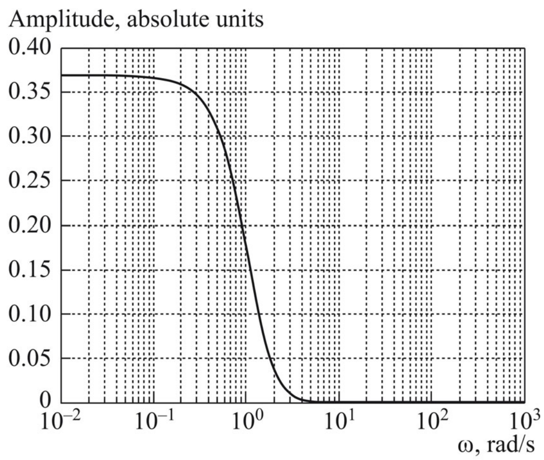

Fig. 3.

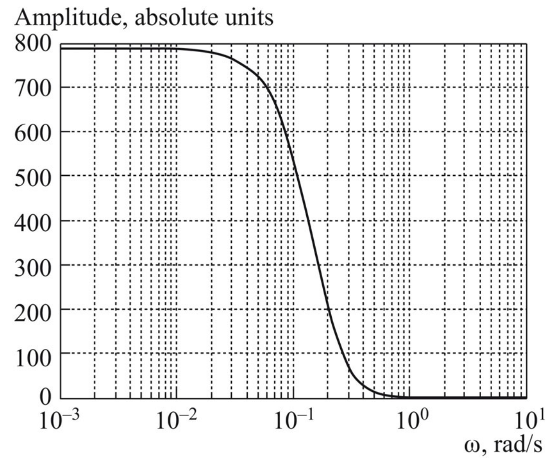

Fig. 4.

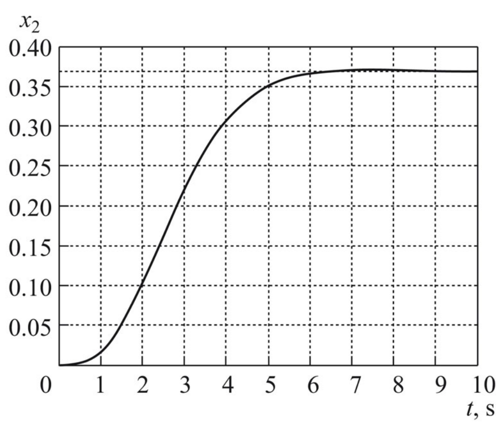

Fig. 5.

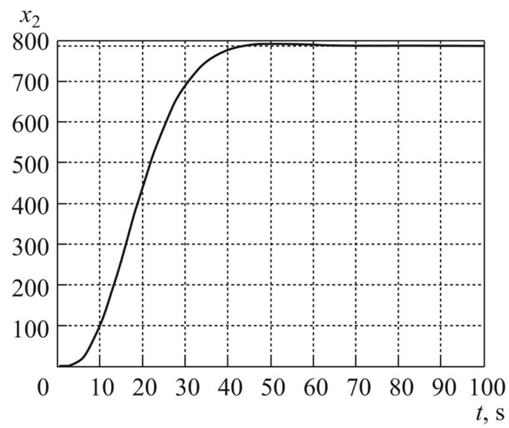

Fig. 6.

the method of [1], where the external disturbance is disregarded during synthesis) are monotone decreasing functions of the frequency ω, so the worst possible external disturbance f in both cases is a step function.

Note that initial points on Figs. 3 and 4 do not depend on the value of parameter q, which is a direct consequence of the fact that the transfer coefficient of object, which equals 0.5, does not depend on q. Figure 5 shows the step response in the closed-loop system with respect to the controlled variable $y = x_2$ for zero initial conditions and a unit step function as f. For comparison, Fig. 6 shows the step response with a controller from [1], where the external disturbance is disregarded during synthesis.

Analyzing the step response, we see that simply accounting in the extended vector of controlled variables z for the weighted signal y in the extended synthesis procedure lets us decrease the steady state error in the closed-loop system by three orders of magnitude as compared to [1]. The reason for this is the fact that the object is minimal phase (it does not have zeros at all [1], and its transfer function is $q/(s^4 + 2qs^2)$). This lets us achieve arbitrarily small control errors as we increase the weight coefficient q_1 based on (5.8) under rather wide margins on q. Guaranteed margin boundaries on the varying parameter with nominal value 0.8 have been found with formulas (4.3). Since parameter q is the multiplier in the transfer function of the open system (4.1), then, having found intersection points of the Nyquist diagram with the real axis (they are -3.756 and -0.093), Results of system analysis

	Disregarding external disturbance [1]	Accounting for disturbance ($q_1 = 1$)
Transfer function of the synthesized controller	$\frac{-1255s^3+8016s^2-112.1s-5.679}{s^4+17.12s^3+195s^2+1178s+4470}$	$\frac{-3.110^8 s^3-4.2\times10^8 s^2-4.6\times10^8 s-1.6\times10^8}{s^4+193.7s^3+2.2\times10^4 s^2+1.4\times10^6 s+6.1\times10^7}$
Value of parameter γ	1.1701	1.1306
Guaranteed radius of stability margins r	0.8546	0.8845
Guaranteed robust stability boundaries with respect to parameter q	$0.4314 < q < 5.5016$	$0.4245 < q < 6.9264$
True robust stability boundaries with respect to parameter q	$0.1985 < q < 8.8594$	$0.213 < q < 8.6022$
Gain and phase stability margins after breaking the system with respect to plant input/output	$L = 0.57$ dB, $\phi_c = 6.63^\circ$	$L = 20.7$ dB, $\phi_c = 65.2^\circ$

we find true stability boundaries for the varying parameter q. The table summarizes the results of our study of the object (7.1) closed by the resulting controller and a controller from [1]. Note that true and guaranteed margins for the parameter q significantly exceed the margins of known synthesis methods [12, 13]: $0.4459 < q < 2.066$.

The last row of the table shows stability gain L and phase φ_c stability margins which are widely used in engineering practice and which are found by opening the closed-loop system with respect to variable u (physical plant input) or y (physical output) [3, 4, 20]. We note rather low stability margins with respect to both phase and amplitude (the stability margin radius at these opening points is very small and equals 0.063) if the controller is constructed with no account for external disturbance, although the bounds on the varying parameter are rather wide. Note that Example 1 of the work [20], which uses the same plant, presents a completely opposite situation: the radius of stability margins at the input (output) of the plant is significant, while a small deviation of parameter q from the nominal value leads to instability. Thus, it becomes obvious that we need a method for robust controller synthesis that would account not only for possible finite deviations of physical parameters from nominal values but also a given radius of stability margins at the physical input (output) of the plant. This, however, remains an plant of further study.

8. CONCLUSION

In this work, we have presented one possible solution for the robust stabilization problem under parametric uncertainty and under the action of external power-bounded polyharmonic disturbances. We note several advantages of the proposed approach as compared to known ones.

1. We consider deviations of *physical* parameters from nominal values.

2. The synthesis procedure reduces to a standard H_∞-control problem to solve which we have readily available and widely used software [12].

3. Clear engineering meaning of synthesis criteria (radii of stability margins and steady state), non-iterative nature and simplicity of the synthesis procedure, as opposed to the widely acclaimed μ-synthesis procedure [4, 12–15].

4. The order of a controller obtained after solving the synthesis problem does not exceed the order of the physical control plant, which is important for practical applications.

Remark 1. Generally speaking, amplitudes and frequencies of the external disturbance (2.4) are fixed only on an interval that does not exceed the settling time in the closed-loop system (2.1), (2.7). This follows from the lemma on steady-state values [6, 17] for matrix frequency inequalities (5.3)–(5.6) and the disturbance property (2.5) independent on the frequencies of external disturbances.

Remark 2. The circular absolute stability criterion [21] (the objective inequality (4.2) is one of its versions) implies that deviations of parameters from nominal values in the boundaries (4.3) may be nonstationary. Besides, a part of or all circuits with respect to variables $\widetilde{u}_i$ $(i = \overline{1,n})$ on Fig. 1 may contain nonstationary nonlinear elements from the class $(1/(1+r), 1/(1-r))$ (which belongs to the Hurwitz angle, a sector) that do not violate asymptotic stability of system on Fig. 1 as a whole.

One drawback of this approach is the sufficiency of estimates (4.3) and (2.8), (5.9). Certain sufficiency is introduced by passing from the auxiliary inequality (5.3) to objective inequalities (5.4). However, our analysis of step response and margins on the parameter in the example of Section 7 indicate that the degree of sufficiency of all these estimates is not too high.

APPENDIX

Proof of Theorem 1. To be definite, assume that the varying physical parameters of the system are: k arbitrary elements of matrices $L_1^{(\beta_1)}$ $(\beta_1 = \overline{1,\alpha_1})$: $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_{k+q}$, where $k + q = n$ is the total number of varying parameters. We introduce the following notation:

$$L_{10}(p) = L_1(p)|_{\lambda_i=0}, \quad i = \overline{1,k}, \quad L_{20}(p) = L_2(p)|_{\lambda_{k+j}=0}, \quad j = \overline{1,q},$$

where in the corresponding matrices the varying parameters are replaced by zeros. We represent the original plant (2.1) in an equivalent form (where we have omitted the arguments to simplify notation and denoted $z_f = z$):

$$\begin{gathered} L_{10}z = L_{20}u + L_3 f + \sum_{r=1}^{k} e_{i_r}\widetilde{u}_r + \sum_{r=k+1}^{n} e_{\alpha_r}\widetilde{u}_r, \\ \widetilde{u}_r(p) = -\lambda_r p^{(\beta_1)_r} e_{j_r}^{\mathrm{T}} z, \quad r = \overline{1,k}, \\ \widetilde{u}_r(p) = \lambda_r p^{(\beta_2)_r} \eta_{\beta_r}^{\mathrm{T}} u, \quad r = \overline{k+1,n}, \\ y = Nz. \end{gathered} \tag{A.1}$$

Here $(\beta_s)_r$ $(s = 1, 2)$ is the index of matrix $L_s^{(\beta_s)}$ where parameter λ_r is located $(r = \overline{1,n})$; (i_r, j_r) are the row and column of matrix $L_1^{(\beta_1)}$ where parameter λ_r is located for $r = \overline{1,k}$; (α_r, β_r) are the row and column of matrix $L_2^{(\beta_2)}$, where parameter λ_r is located for $r = \overline{k+1,n}$; e_{j_r}, η_{β_r} are vectors of size l and m respectively:

$$e_{j_r} = \begin{cases} e_{j_r,i} = 0 & \text{for} \quad i \neq j_r \\ e_{j_r,j_r} = 1, \end{cases} \qquad \eta_{\beta_r} = \begin{cases} \eta_{\beta_r,i} = 0 & \text{for} \quad i \neq \beta_r \\ \eta_{\beta_r,\beta_r} = 1. \end{cases}$$

We introduce the matrix E that unites vectors e_{i_r} and e_{α_r} from the first relation of (A.1) and, constructing the n-dimensional vector of dummy inputs $\widetilde{u} = [\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_{k+1}, \ldots, \widetilde{u}_n]^{\mathrm{T}}$, after passing to the Laplace transform under zero initial conditions we get from this relation that

$$z = (L_{10}(s))^{-1}E\widetilde{u} + (L_{10}(s))^{-1}L_{20}(s)u + (L_{10}(s))^{-1}L_3(s)f. \tag{A.2}$$

Taking into account (A.2), the last equation of (A.1), and the second relation from (3.1), we conclude that

$$W_{21} = N(L_{10}(s))^{-1}E, \quad W_{22} = N(L_{10}(s))^{-1}L_{20}(s), \quad W_{23} = N(L_{10}(s))^{-1}L_3(s). \tag{A.3}$$

We now obtain expressions for n dummy outputs of the plant $\widetilde{y}_r$ united into the vector $\widetilde{y}$. To satisfy $\widetilde{u} = \Lambda\widetilde{y}$, dummy outputs in (A.1) must have the form $\widetilde{y}_r(s) = -s^{(\beta_1)_r} e_{j_r}^{\mathrm{T}} z$ for $(r = \overline{1,k})$ and $\widetilde{y}_r(s) = s^{(\beta_2)_r} \eta_{\beta_r}^{\mathrm{T}} u$ for $(r = \overline{k+1,n})$, and the matrix from (3.1) W_{1s} $(s = \overline{1,3})$ will have the following form:

$$W_{11}(p) = \begin{pmatrix} -s^{(\beta_1)_r} e_{j_r}^{\mathrm{T}}, \quad r = \overline{1,k} \\ 0, \quad q \times l \end{pmatrix} (L_{10}(s))^{-1} E,$$

$$W_{12}(s) = \begin{pmatrix} -s^{(\beta_1)_r} e_{j_r}^{m} (L_{10}(s))^{-1} L_{20}(s), \quad r = \overline{1,k} \\ s^{(\beta_2)_r} \eta_{\beta_r}^{m}, \quad r = \overline{k+1,n} \end{pmatrix}, \tag{A.4}$$

$$W_{13}(s) = \begin{pmatrix} -s^{(\beta_1)_r} e_{j_r}^{m}, \quad r = \overline{1,k} \\ 0, \quad q \times l \end{pmatrix} (L_{10}(s))^{-1} L_3(s),$$

as required. Note that relations (A.2)–(A.4) are meaningful if matrix $L_{10}(s) = L_1(s)|_{\lambda_i=0}$ $(i = \overline{1,k})$ is nonsingular. Otherwise we can take some λ_i to be nonzero, and $\lambda_i = \varepsilon_i$, where ε_i are sufficiently small compared to the new nominal $\lambda_i - \varepsilon_i$.

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