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DETERMINATE SYSTEMS

Finite-Frequency Identification: Test Frequency Bounds[1]

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Abstract—A test harmonic signal underlies the finite-frequency identification method. It not easy to specify its frequencies, because they must be chosen in the range of eigenfrequencies determined by the coefficients of the identified object. A method of determining the eigenfrequency bounds for the identified object is developed.

1. INTRODUCTION

In identification of linear stationary objects, there are two trends differing in assumptions concerning disturbances and measurement noises.

In the first trend [1], disturbances and measurement noises are assumed to be random processes with known distribution and statistical properties. In the second trend [2–4], they are unknown bounded functions. Algorithms of this trend employ the output of the object, which contains two indeterminate components: the first depends on the unknown coefficients of the object and the second depends on these coefficients and unknown disturbance. As a result of the unknown disturbance, identification accuracy is bounded and depends on disturbance.

The finite-frequency identification method [5, 6] is based on a test signal, which aids in discerning these components and obtaining the necessary identification accuracy if the disturbance satisfies experimentally verifiable ff-filtrability conditions [6].

The traditional frequency approach is also based on test signals. But here the error in the measurement of frequency characteristics is assumed to be absent [7] or random numbers [8] uncorrelated at certain test frequencies. Therefore, it is difficult to determine the class of disturbances and measurement noises for which this approach yields the desired identification accuracy.

The traditional frequency approach requires the use of many test frequencies. The number of test frequencies required in the finite-frequency method is minimal and equal to the dimension of the state space of the object; therefore, there is a need for choosing test frequencies. Intuitively, it is obvious that they must be chosen in the range of frequencies at which the logarithmic amplitude-frequency characteristic of the object has kinks. These kinks depend on the unknown coefficients of the object. Therefore, there is a need for determining the bounds of this range. In this paper, we study the determination of these bounds.

2. THE FINITE-FREQUENCY IDENTIFICATION METHOD. STABLE OBJECTS

Let us consider a completely controllable and asymptotically stable object described by the equation

$$d_n y^{(n)} + \cdots + d_1 \dot{y} + y = k_\gamma u^{(\gamma)} + \cdots + k_1 \dot{u} + k_0 u + f, \quad t \ge t_0, \tag{1}$$

[1] This paper is an enlarged variant of the report [12] read at the 3rd Asian Control Conference at Shanghai, China.

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where $y(t)$ is the measured output, $u(t)$ is the controlled input, $y^{(i)}$ and $u^{(j)}$ $(i = \overline{1,n},\ j = \overline{1,\gamma})$ are the derivatives of these functions, and $f(t)$ is an unknown bounded disturbance. The coefficients d_i and k_j $(i = \overline{1,n},\ j = \overline{0,\gamma})$ are unknown, but n and γ, $\gamma < n$, are known.

Identification consists in finding the estimates $\hat{d}_i$ and $\hat{k}_j$ $(i = \overline{1,n}, j = \overline{0,\gamma})$ of coefficients such that the identification errors $\Delta d_i = d_i - \hat{d}_i$ and $\Delta k_i = k_i - \hat{k}_i$ satisfy the conditions

$$|\Delta d_i| \le \varepsilon_i^d, \quad |\Delta k_j| \le \varepsilon_j^k \quad \left(i = \overline{1,n},\ j = \overline{0,\gamma}\right), \tag{2}$$

where ε_i^d and ε_j^k $(i = \overline{1,n},\ j = \overline{0,\gamma})$ are given numbers.

We now describe the finite-frequency method of identification [5, 6] for solving this problem.

The $2n$ numbers

$$\alpha_k = \operatorname{Re} w(j\omega_k), \quad \beta_k = \operatorname{Im} w(j\omega_k) \quad \left(k = \overline{1,n}\right), \tag{3}$$

where

$$w(s) = \frac{k_\gamma s^\gamma + \cdots + k_1 s + k_0}{d_n s^n + \cdots + d_1 s + 1}, \tag{4}$$

are called the frequency parameters [9]. Their estimates are experimentally determined as follows: object (1) is excited by a test signal

$$u = \sum_{k=1}^{n} \rho_k \sin \omega_k (t - t_0), \quad t \ge t_0, \tag{5}$$

of given positive amplitudes ρ_k $\left(k = \overline{1,n}\right)$ and test frequencies ω_k $\left(k = \overline{1,n}\right)$. The test frequencies are multiples of the base frequency $\omega_b : \omega_i = n_i \omega_b$, where n_i $\left(i = \overline{1,n}\right)$ are integers. The output of the object excited by the test signal (5) is applied to a Fourier filter

$$\hat{\alpha}_k = \alpha_k(\tau) = \frac{2}{\rho_k \tau} \int\limits_{t_0}^{t_0+\tau} y(t) \sin \omega_k (t - t_0) dt, \tag{6}$$

$$\hat{\beta}_k = \beta_k(\tau) = \frac{2}{\rho_k \tau} \int\limits_{t_0}^{t_0+\tau} y(t) \cos \omega_k (t - t_0) dt \quad \left(k = \overline{1,n}\right),$$

where τ is the filtering time, which is a multiple of the base period $T_b = \dfrac{2\pi}{\omega_b}$.

To formulate conditions for the convergence of the estimates $\alpha_k(\tau)$ and $\beta_k(\tau)$ $\left(k = \overline{1,n}\right)$ to their true values, let us introduce the filtrability functions

$$\ell_k^\alpha(\tau) = \frac{2}{\rho_k \tau} \int\limits_{t_0}^{t_0+\tau} \bar{y}(t) \sin \omega_k (t - t_0)\, dt, \tag{7}$$

$$\ell_k^\beta(\tau) = \frac{2}{\rho_k \tau} \int\limits_{t_0}^{t_0+\tau} \bar{y}(t) \cos \omega_k (t - t_0)\, dt \quad \left(k = \overline{1,n}\right),$$

where $\bar{y}(t)$ is the output of the object when there is no test signal (5) $(u(t) = 0)$.

Assertion 1 ([5]). *If the disturbance $f(t)$ is such that the conditions*

$$|\ell_k^\alpha(\tau)| \le \varepsilon_k^\alpha, \quad \left|\ell_k^\beta(\tau)\right| \le \varepsilon_k^\beta \quad \left(k = \overline{1,n}\right), \quad \tau \ge \tau^*, \tag{8}$$

are satisfied beginning from a certain instant $\tau = \tau^*$, *where* ε_k^α *and* ε_k^β $(k = \overline{1,n})$ *are given numbers, then there exists a filtering instant* $\tau = \bar{\tau}^* \geq \tau^*$ *such that the filtering errors* $\Delta\alpha_k(\tau) = \hat{\alpha}_k - \alpha_k$ *and* $\Delta\beta_k(\tau) = \hat{\beta}_k - \beta_k$ *satisfy the inequalities*

$$|\Delta\alpha_k(\tau)| \leq \varepsilon_k^\alpha, \quad |\Delta\beta_k(\tau)| \leq \varepsilon_k^\beta \quad \left(k = \overline{1,n}\right), \quad \tau \geq \bar{\tau}^*. \tag{9}$$

A disturbance satisfying conditions (8) is said to be *ff-filtrable* [5].

A disturbance is said to be *strictly ff-filtrable* if $\lim\limits_{\tau\to\infty} \ell_k^\alpha(\tau) = \lim\limits_{\tau\to\infty} \ell_k^\beta(\tau) = 0$ $\left(k = \overline{1,n}\right)$. In this case, the filtering errors have the property

$$\lim_{\tau\to\infty} \Delta\alpha_k(\tau) = \lim_{\tau\to\infty} \Delta\beta_k(\tau) = 0. \tag{10}$$

Using frequency parameter estimates, we can estimate the coefficients of the object. Indeed, the identity $w(s) = \dfrac{k(s)}{d(s)}$ and expressions (3) yield the system of linear algebraic equations

$$\hat{k}(s_k) - (\alpha_k + j\beta_k)\hat{\bar{d}}(s_k) = \alpha_k + j\beta_k \quad \left(k = \overline{1,n}\right), \tag{11}$$

where $\hat{\bar{d}}(s) = \hat{d}(s) - 1 = \hat{d}_n s^n + \cdots + \hat{d}_1 s$, $\hat{k}(s) = \hat{k}_\gamma s^\gamma + \cdots + \hat{k}_0$, $s_k = j\omega_k$ $\left(k = \overline{1,n}\right)$.

Assertion 2 ([9]). *If object* (1) *is completely controllable, then system* (11) *has a unique solution* $\hat{d}_i = d_i$, $\hat{k}_j = k_j$ $\left(i = \overline{1,n},\ j = \overline{0,\gamma}\right)$, *which does not depend on the test frequencies* ω_i $[\omega_i \neq \omega_j$ $(i \neq j)$, $\omega_i \neq 0$ $\left(i = \overline{1,n}\right)]$.

In Eqs. (11), replacing the frequency parameters α_k and β_k $\left(k = \overline{1,n}\right)$ by their estimates, we obtain the frequency identification equations

$$\hat{k}(s_k) - (\hat{\alpha}_k + j\hat{\beta}_k)\hat{\bar{d}}(s_k) = \hat{\alpha}_k + j\hat{\beta}_k \quad \left(k = \overline{1,n}\right). \tag{12}$$

The identification time (filtering duration) $\tau = qT_b$ $(q = 1, 2, \ldots)$ is determined from the following necessary conditions for the convergence of identification:

$$\begin{matrix} |d_i(qT_b) - d_i[(q-1)T_b]| \leq \varepsilon_i^d |d_i(qT_b)| & \left(i = \overline{1,n}\right) \\ |k_j(qT_b) - k_j[(q-1)T_b]| \leq \varepsilon_j^k |k_j(qT_b)| & \left(j = \overline{0,\gamma}\right) \end{matrix} \quad (q = 1, 2, \ldots). \tag{13}$$

Algorithm 1 (of finite-frequency identification) [5] consists of the following steps:

(a) apply the output of object (1) excited by the test signal (5) to the input of the Fourier filter (6),

(b) measure the outputs of the Fourier filter at instants qT_b $(q = 1, 2, \ldots)$,

(c) for every $\tau = qT_b$ $(q = 1, 2 \ldots)$, solve the frequency Eqs. (12), where $\hat{\alpha}_k = \alpha_k(qT_b)$ and $\hat{\beta}_k = \beta_k(qT_b)$ $\left(k = \overline{1,n}\right)$, and find the estimates $d_i(qT_b)$ and $k_j(qT_b)$ $\left(i = \overline{1,n},\ j = \overline{0,\gamma}\right)$ of the coefficients of the object, and

(d) verify the necessary conditions (13) for every q until they are satisfied for some $q = q_1$.

Whether the estimates $d_i(q_1T_b)$ and $k_j(q_1T_b)$ $\left(i = \overline{1,n},\ j = \overline{0,\gamma}\right)$ satisfy conditions (2) is verified by model validation frequency methods [5].

3. FORMULATION OF THE PROBLEM

In the identification algorithm described above, the amplitudes and frequencies of the test signal (5) are assumed to be given. In reality, they are unknown and are determined in the first identification stage—the experiment design stage [1].

If the test frequencies are known, then the amplitudes of the test signal are determined from the "small disturbance" condition [6], which implies that the test action must not strongly change the natural output $\bar{y}(t)$ of the object. This condition is of the form

$$|y(t) - \bar{y}(t)| \leq \varepsilon_y, \quad t \geq t_0, \tag{14}$$

where ε_y is a given number.

It is not a simple matter to find the test frequencies. Intuitively, it is obvious that they must be chosen in the frequency range in which the logarithmic amplitude-frequency (LAF) characteristic of the object has kinks. Such a conclusion is not consistent with Assertion 2, which asserts that the solution of Eq. (11) does not depend on test frequencies. This is an apparent contradiction, because the frequency identification Eqs. (12) derived from system (11) are based on the use of estimates of frequency parameters, instead of their true values. Moreover, we can prove that for test frequencies chosen outside the range of kinks of the LAF curve, the identification accuracy condition (2) may be violated for arbitrarily small filtering errors.

Using a more exact description of the choice of test frequencies, let us represent the transfer function of object (1), assuming that $k_0 \neq 0$, as

$$w(s) = k\frac{\prod\limits_{i=1}^{p_3}(\check{T}_i s + 1)\prod\limits_{i=1}^{p_4}(\check{\check{T}}_i^2 s^2 + 2\check{\check{T}}_i \check{\check{\xi}}_i s + 1)}{\prod\limits_{i=1}^{p_1}(\bar{T}_i s + 1)\prod\limits_{i=1}^{p_2}(\bar{\bar{T}}_i^2 s^2 + 2\bar{\bar{T}}_i \check{\check{\xi}}_i s + 1)}, \quad \check{\check{\xi}}_i < 1(i = \overline{1, p_4}), \quad \check{\check{\xi}}_i < 1(1 = \overline{1, p_2}). \tag{15}$$

Let the time constants of these functions be arranged in decreasing order

$$\begin{gathered} \bar{T}_1 > \bar{T}_2 > \cdots > \bar{T}_{p_1}, \quad \bar{\bar{T}}_1 > \bar{\bar{T}}_2 > \cdots > \bar{\bar{T}}_{p_2}, \\ |\check{T}_1| > |\check{T}_2| > \cdots > |\check{T}_{p_3}|, \quad |\check{\check{T}}_1| > |\check{\check{T}}_2| > \cdots > |\check{\check{T}}_{p_4}|. \end{gathered} \tag{16}$$

Definition 1. Eigenfrequencies of the object are defined to be the numbers

$$\bar{\omega}_i = \frac{1}{\bar{T}_i}\left(i = \overline{1, p_1}\right), \quad \bar{\bar{\omega}}_i = \frac{1}{\bar{\bar{T}}_i}\left(i = \overline{1, p_2}\right), \quad \check{\omega}_i = \frac{1}{|\check{T}_i|}\left(i = \overline{1, p_3}\right), \quad \check{\check{\omega}}_i = \frac{1}{|\check{\check{T}}_i|}\left(i = \overline{1, p_4}\right). \tag{17}$$

The eigenfrequencies are the frequencies at which kinks occur in the LAF curve.

The frequencies $\omega_\ell = \min\left\{\bar{\omega}_1, \bar{\bar{\omega}}_1, \check{\omega}_1, \check{\check{\omega}}_1\right\}$ and $\omega_{\text{u}} = \max\left\{\bar{\omega}_{p_1}, \bar{\bar{\omega}}_{p_2}, \check{\omega}_{p_3}, \check{\check{\omega}}_{p_4}\right\}$ are called the *lower* and *upper bounds* of eigenfrequencies and the interval $[\omega_\ell, \omega_{\text{u}}]$, the eigenfrequency range.

By definition, these bounds depend on the greatest and least time constants of the transfer function (15). Therefore, the following cases are possible: $\omega_\ell = \bar{T}_1^{-1}$, $\omega_\ell = \bar{\bar{T}}_1^{-1}$, $\omega_\ell = |\check{T}_1^{-1}|$, $\omega_\ell = |\check{\check{T}}_1^{-1}|$, $\omega_{\text{u}} = \bar{T}_{p_1}^{-1}$, $\omega_{\text{u}} = \bar{\bar{T}}_{p_2}^{-1}$, $\omega_{\text{u}} = |\check{T}_{p_3}^{-1}|$, and $\omega_{\text{u}} = |\check{\check{T}}_{p_4}^{-1}|$.

For the sake of simplicity of presentation, we only consider the most frequently encountered case, in which the eigenfrequency bounds depend on the time constants of inertial components:

$$\omega_\ell = \bar{T}_1^{-1}, \quad \omega_{\text{u}} = \bar{T}_{p_1}^{-1}. \tag{18}$$

Let the time constants of the object satisfy the following constraints

$$\bar{T}_1 \geq |\Delta T|\delta_\ell^{-1}, \quad 0 < \delta_\ell < 0.5, \quad \bar{T}_{p_1}^{-1} \geq |\Delta\omega|\delta_{\text{u}}^{-1}, \quad 0 < \delta_{\text{u}} < 0.5, \tag{19}$$

where

$$\Delta T = \sum_{i=2}^{p_1} \bar{T}_i + \sum_{i=1}^{p_2} 2\, \bar{\tilde{T}}_i \bar{\tilde{\zeta}}_i - \sum_{i=1}^{p_3} \check{T}_i - \sum_{i=1}^{p_4} 2\, \check{\tilde{T}}_i \check{\tilde{\zeta}}_i, \tag{20}$$

$$\Delta\omega = \sum_{i=1}^{p_1-1} (\bar{T}_i)^{-1} + \sum_{i=1}^{p_2} 2\, \bar{\tilde{\zeta}}_i \left(\bar{\tilde{T}}_i \right)^{-1} - \sum_{i=1}^{p_3} \left(\check{T}_i \right)^{-1} - \sum_{i=1}^{p_4} 2\, \check{\tilde{\zeta}}_i \left(\check{\tilde{T}}_i \right)^{-1}. \tag{21}$$

The bounds δ_ℓ^{-1} and $\delta_{\rm u}^{-1}$ serve as a measure of remoteness of the time constants $\bar{T}_1$ and $\bar{T}_{p_1}$ from other time constants of the object.

According to expressions (18), to determine the eigenfrequency bounds, we must identify the time constants $\bar{T}_1$ and $\bar{T}_{p_1}$. Obviously, the identification of these bounds is influenced by other time constants in ΔT and $\Delta\omega$. Therefore, let us introduce lower and upper *pseudo-frequencies*

$$\omega_\ell^n = \frac{1}{T_\ell}, \quad \omega_{\rm u}^n = \bar{T}_{p_1}^{-1} + \Delta\omega, \tag{22}$$

where

$$T_\ell = \bar{T}_1 + \Delta T. \tag{23}$$

It is a simple matter to verify that the lower and upper bounds of eigenfrequencies under condition (19) satisfy the inequalities

$$\omega_\ell^n (1 - \delta_\ell) \le \omega_\ell \le \omega_\ell^n (1 + \delta_\ell), \quad \omega_{\rm u}^n (1 - \delta_{\rm u})^{-1} \ge \omega_{\rm u} \ge \omega_{\rm u}^n (1 + \delta_{\rm u})^{-1}. \tag{24}$$

These inequalities imply that the pseudo-frequencies are close to the unknown bounds even in the most unfavorable case $\delta_\ell = \delta_{\rm u} = 0.5$. Therefore, the determination of these bounds under conditions (19) is reduced to determining the pseudo-frequencies.

Problem 1. Find the estimates $\hat{\omega}_\ell^n$ and $\hat{\omega}_{\rm u}^n$ of the lower and upper pseudo-frequencies such that the errors $\Delta\omega_\ell^n = \omega_\ell^n - \hat{\omega}_\ell^n$, $\Delta\omega_{\rm u}^n = \omega_{\rm u}^n - \hat{\omega}_{\rm u}^n$ satisfy the conditions

$$|\Delta\omega_\ell^n| \le \varepsilon_\ell^n, \quad |\Delta\omega_{\rm u}^n| \le \varepsilon_{\rm u}^n, \tag{25}$$

where ε_ℓ^n and $\varepsilon_{\rm u}^n$ are given numbers.

4. THE LOWER BOUND OF TEST FREQUENCIES

The lower bound is determined as follows: the transfer function of object (1) is expressed as $w(s) = \dfrac{1}{\bar{T}_1 s + 1} w_{un}(s)$, where $w_{un}(s)$ is a transfer function that contains other time constants and the coefficient k. If object (1) is excited by a test signal (5) with $n = 1$ and small frequency ω_1 ($\omega_1 < \omega_\ell$), then the output is almost exactly described by the equation

$$\hat{\bar{T}}_1 \dot{y} + y = k_m u + f. \tag{26}$$

The frequency Eq. (12) for the identification of model (26) is of the form

$$k_m - \left(\hat{\alpha}_1 + j\hat{\beta}_1 \right) \hat{\bar{T}}_1 j\omega_1 = \hat{\alpha}_1 + j\hat{\beta}_1. \tag{27}$$

This equation can be written as a system $k_m + \omega_1 \hat{\beta}_1 \hat{\bar{T}}_1 = \hat{\alpha}_1$, $-\omega_1 \hat{\alpha}_1 \hat{\bar{T}}_1 = \hat{\beta}_1$, and the second equation of this system defines the function

$$\bar{T}_1(\omega_1, \tau) = -\frac{\beta_1(\tau)}{\omega_1 \alpha_1(\tau)}. \tag{28}$$

The estimate ω_ℓ^n is determined through the following operations.

Experiment 1. Object (1) is excited by a test signal

$$u(t) = \rho_1 \sin \omega_1 (t - t_0), \tag{29}$$

where $\omega_1 = \omega_1^{[1]}$ is a sufficiently small number. The output of the object is applied to the input of the Fourier filter (6) for $n = 1$ and then function (28) is computed.

Assertion 3. *Function* (28) *has the structure*

$$\bar{T}_1(\omega_1, \tau) = T_\ell + \varepsilon_\ell(\omega_1, \tau), \tag{30}$$

where the function $\varepsilon_\ell(\omega_1, \tau)$ is such that, if the disturbance $f(t)$ is strictly ff-filtrable, then, for any given arbitrarily small ε_ℓ^*, there exist a small frequency ω_1 and a large filtering time τ^* for which

$$|\varepsilon_\ell(\omega_1, \tau)| \le \varepsilon_\ell^*, \quad \tau \ge \tau^*. \tag{31}$$

If $f(t)$ is simply ff-filtrable, then the number ε_ℓ^* depends on the numbers ε_k^α and ε_k^β $(k = \overline{1, n})$.

The proof of this assertion is given in the Appendix.

According to this assertion, the unknown estimate of the lower pseudo-frequency is

$$\hat{\omega}_\ell^n = \bar{T}_1^{-1}(\omega_1, \tau^*), \tag{32}$$

and it satisfies the first condition in (25).

To verify whether ω_1 is small, experiment 1 is repeated with $\omega_1 = \omega_1^{[2]} = \omega_1^{[1]}/2$ and then the number $\bar{T}_1(\omega_1^{[2]}, \tau^*)$ is computed. If

$$\left|\bar{T}_1\left(\omega_1^{[1]}, \tau^*\right) - \bar{T}_1\left(\omega_1^{[2]}, \tau^*\right)\right| \le \delta_T \left|\bar{T}_1\left(\omega_1^{[2]}, \tau^*\right)\right|, \tag{33}$$

where δ_T is a sufficiently small given number, then $\hat{\omega}_\ell^n = \left[\bar{T}_1\left(\omega_1^{[2]}, \tau^*\right)\right]^{-1}$. If conditions (33) are not satisfied, then the experiment is repeated with $\omega_1 = \omega_1^{[3]} = \omega_1^{[2]}/2$, etc. These experiments must be accompanied with a verification of the ff-filtrability conditions (8) for $n = 1$, and the amplitude ρ_1 is determined from the "small disturbance" condition (14).

5. THE UPPER BOUND OF TEST FREQUENCIES

If object (1) is excited by signal (29) with a sufficiently high frequency ω_1 $(\omega_1 > \omega_{\mathrm{u}})$, then its output is almost exactly described by the equation

$$\hat{\bar{T}}_{p_1} y^{(n-\gamma)} + y^{(n-\gamma-1)} = k_m u + f. \tag{34}$$

The frequency Eqs. (12) for identifying model (34) are of the form

$$k_m - \left(\hat{\alpha}_1 + j\hat{\beta}_1\right)(j\omega_1)^{n-\gamma} \hat{\bar{T}}_{p_1} = \left(\hat{\alpha}_1 + j\hat{\beta}_1\right)(j\omega_1)^{n-\gamma-1}. \tag{35}$$

Hence we obtain the formulas

$$\bar{T}_{p_1}(\omega_1, \tau) = \frac{\alpha_1(\tau)}{\omega_1 \beta_1(\tau)} \text{ for even } (n - \gamma), \tag{36}$$

$$\bar{T}_{p_1}^a(\omega_1, \tau) = -\frac{\beta_1(\tau)}{\omega_1 \alpha_1(\tau)} \text{ for odd } (n - \gamma). \tag{37}$$

Experiment 2. Exciting object (1) by a test signal (29), where $\omega_1 = \omega_1^{[1]}$ is a sufficiently large number, apply the output to the Fourier filter (6) for $n = 1$ and compute function (36) or (37).

Assertion 4. *The functions* $\bar{T}_{p_1}(\omega_1,\tau)$ *and* $\bar{T}^a_{p_1}(\omega_1,\tau)$ *have the structure*

$$\bar{T}_{p_1}(\omega_1,\tau) = (\omega_{\rm u}^n)^{-1} + \varepsilon_{\rm u}(\omega_1,\tau), \quad \bar{T}^a_{p_1}(\omega_1,\tau) = (\omega_{\rm u}^n)^{-1} + \bar{\varepsilon}_{\rm u}(\omega_1,\tau), \tag{38}$$

where the functions $\varepsilon_{\rm u}(\omega_1,\tau)$ *and* $\bar{\varepsilon}_{\rm u}(\omega_1,\tau)$ *are such that if the disturbance* $f(t)$ *is strictly ff-filtrable, then for any given arbitrarily small* $\varepsilon_{\rm u}^*$ *there exist a sufficiently large frequency* ω_1 *and a filtering time* τ^* *for which*

$$|\varepsilon_{\rm u}(\omega_1,\tau)| \le \varepsilon_{\rm u}^*, \quad |\bar{\varepsilon}_{\rm u}(\omega_1,\tau)| \le \varepsilon_{\rm u}^*, \quad \tau \ge \tau^*. \tag{39}$$

If $f(t)$ *is simply ff-filtrable, then the number* $\varepsilon_{\rm u}^*$ *depends on the numbers* ε_k^α *and* ε_k^β $(k=\overline{1,n})$.

The proof of Assertion 4 is given in the Appendix.

From Assertion 4 we obtain the estimate for the upper bound of the pseudo-frequency:

$$\hat{\omega}_{\rm u}^n = \left[\bar{T}_{p_1}(\omega_1,\tau^*)\right]^{-1} \quad \text{or} \quad \hat{\omega}_{\rm u}^n = \left[\bar{T}^a_{p_1}(\omega_1,\tau^*)\right]^{-1}. \tag{40}$$

To verify whether the frequency ω_1 is sufficiently high, experiment 2 is repeated with $\omega_1 = \omega_1^{[2]} = 2\omega_1^{[1]}$, and then numbers (36) or (37) are compared with the numbers obtained for $\omega_1 = \omega_1^{[2]}$. Conditions (8) and (14) are also to be verified in these experiments.

6. UNSTABLE OBJECTS AND MEASUREMENT NOISES

Let us consider an unstable object (1). The positive number $s^* = \max\{\operatorname{Re} s_1, \operatorname{Re} s_2, \ldots, \operatorname{Re} s_n\}$, where s_i $(i=\overline{1,n})$ are the roots of the polynomial $d(s)$, is called the *degree of unstability* of the object. Let its upper estimate C_0 be known. It can be determined experimentally.

Let us introduce a positive number $\lambda \ge C_0$ $(C_0 > s^* > 0)$ and design a device, called the $(-\lambda)$-block, which multiplies the output $y(t)$ of the object by the function $e^{-\lambda(t-t_0)}$. Its output is

$$\tilde{y}(t) = y(t)e^{-\lambda(t-t_0)}. \tag{41}$$

Obviously, an object with $(-\lambda)$-block is asymptotically stable. But, if it is excited with the test signal (5), its response $\tilde{y}(t)$, like the response to the initial conditions, decays $\lim_{t\to\infty}\tilde{y}(t)=0$. Therefore, let us integrate the object with a $(+\lambda)$-block, which multiplies the input signal of (5) by the function $e^{\lambda(t-t_0)}$ and, consequently, this signal can be used as the input for object (1) [10]

$$\tilde{u} = u(t)e^{\lambda(t-t_0)}. \tag{42}$$

Thus, we obtain a system consisting of object (1), a $(-\lambda)$-block, and a $(+\lambda)$-block, whose input, as before, is $u(t)$ and the output is $\tilde{y}(t)$.

Let us determine its transfer function $\tilde{w}(s) = \tilde{y}(s)/u(s)$. For this, divide Eq. (1) by d_n and express it in Cauchy form as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\tilde{u} + \mathbf{m}f, \quad y = \mathbf{c}^{\rm T}\mathbf{x}, \tag{43}$$

where the $n\times n$ matrix A and the n-dimensional column vectors $\mathbf{b}$, $\mathbf{m}$, and $\mathbf{c}$ (T is the transpose) are constructed from the coefficients d_i and k_j $(i=\overline{1,n},\ j=\overline{0,\gamma})$.

Let us introduce a vector $\tilde{\mathbf{x}} = \mathbf{x}e^{-\lambda(t-t_0)}$. Using notation (41) and (42) and the expression $\dot{\mathbf{x}} = \dot{\tilde{\mathbf{x}}}e^{\lambda(t-t_0)} + \tilde{\mathbf{x}}\lambda e^{\lambda(t-t_0)}$, we obtain the equations

$$\dot{\tilde{\mathbf{x}}} = (A-\lambda E)\tilde{\mathbf{x}} + \mathbf{b}u + \mathbf{m}e^{-\lambda(t-t_0)}f, \quad \tilde{y} = \mathbf{c}^{\rm T}\tilde{\mathbf{x}}, \tag{44}$$

where E is a unit matrix.

Its corresponding transfer function is

$$\tilde{w}(s) = \frac{\tilde{y}(s)}{u(s)} = \mathbf{c}^{\mathrm{T}} \left[E(s+\lambda) - A\right]^{-1} \mathbf{b} = w(s+\lambda)$$
$$= \frac{k(s+\lambda)}{d(s+\lambda)} = \frac{\tilde{k}(s)}{\tilde{d}(s)} = \frac{\tilde{k}_\gamma s^\gamma + \cdots + \tilde{k}_1 s + \tilde{k}_0}{s^n + \tilde{d}_{n-1}s^{n-1} + \cdots + \tilde{d}_1 s + \tilde{d}_0}. \tag{45}$$

The roots $\tilde{s}_i$ $(i = \overline{1,n})$ of the polynomial $\tilde{d}(s)$ are left zeros. Indeed, since $d(s_i) = 0$ $(i = \overline{1,n})$, we find that the roots of the polynomial $\tilde{d}(s) = d(s+\lambda)$ are $\tilde{s}_i = -\lambda + s_i$ $(i = \overline{1,n})$. By the definition of the number λ, we have $\operatorname{Re} \tilde{s}_i < 0$.

Thus, the system is an asymptotically stable "object" described by Eq. (44) and having the transfer function (45).

This "object" can be identified (if the test frequencies are given) with algorithm 1 if object (1) in operation (a) is replaced by object (44) and the function $y(t)$ in the Fourier filter (6) is replaced by $\tilde{y}(t)$. Moreover, as shown in [10], filtering errors have property (10). This means that the disturbance $\tilde{f}(t) = e^{-\lambda(t-t_0)}f(t)$ is strictly ff-filtrable. Operation (c) yields the estimates $\tilde{d}_i(qT_b)$ and $\tilde{k}_j(qT_b)$ $(i = \overline{1,n},\ j = \overline{0,\gamma})$. Using these estimates, we can uniquely compute the unknown coefficients $d_i(qT_b)$ and $k_j(qT_b)$ $(i = \overline{1,n},\ j = \overline{0,\gamma})$ from formulas derived from the equalities $\tilde{d}(s) = d(s+\lambda)$ and $\tilde{k}(s) = k(s+\lambda)$.

Passing to Problem 1 for unstable objects, we can formally refine Definition 1 of the eigenfrequencies of an object, replacing the time constants $\tilde{T}_i$ and $\tilde{\bar{T}}_j$ $(i = \overline{1,p_1},\ j = \overline{1,p_2})$ in (17) by their moduli. But, since the coefficients of the transfer function (45) are identified (through the solution of frequency Eqs. (12)), Definition 1 must pertain to this transfer function.

The estimates of the lower and upper bounds of the eigenfrequencies of "object" (44) are determined from formulas (32) and (40) with the use of the results of experiments 1 and 2, in which object (1) is replaced by "object" (44) and the Fourier filter (6) $y(t)$ is replaced by $\tilde{y}(t)$.

Remark 1. Sometimes it is difficult to solve problem 1 for unstable objects due to the constraints

$$|\tilde{u}(t)| \leq \tilde{u}^*, \quad |y(t)| \leq y^*, \tag{46}$$

where $\tilde{u}^*$ and y^* are given numbers characterizing the admissible inputs and outputs of object (1). Inequalities (46) restrict the duration of experiments 1 and 2, and the time τ^* in expressions (32) and (40) may not be attainable.

We now consider the case in which the output of object (1) is measured with noise. In this case, Eqs. (1) are supplemented with the expression

$$\check{y} = y + \eta,$$

where $\check{y}(t)$ is the measured output and $\eta(t)$ is the measurement noise, which, like the disturbance $f(t)$, is an unknown bounded function ($|\eta(t)| \leq \eta^*$, where η^* is a number).

It is easy to show that algorithm 1 and the solution of problem 1 remain unchanged even if $y(t)$ and $\bar{y}(t)$ in integrals (6) and (7) are replaced by $\check{y}(t)$ and $\bar{y}(t) + \eta(t)$, respectively.

7. AN EXAMPLE

Let us consider a completely controllable asymptotically stable object described by the equation

$$d_3 \dddot{y} + d_2 \ddot{y} + d_1 \dot{y} + y = k_1 \dot{u} + k_0 u + f \tag{47}$$

with unknown coefficients and unknown bounded disturbance.

The problem here is to estimate the eigenfrequencies.

Remark 2. The tested object (47) is described by the equation

$$0.2\,\dddot{y} + 1.24\,\ddot{y} + 5.24\,\dot{y} + y = -0.4\,\dot{u} + u + f, \tag{48}$$

where the disturbance is described by

$$f(t) = a \operatorname{sgn}\left[\sin \omega^f t\right], \tag{49}$$

$a = 5$, and ω^f is a random number in the interval $[3, 5]$, which changes in every period by $\dfrac{2\pi}{\omega^f}$.

The transfer function of object (48) is

$$w(s) = \frac{25(-0.4s + 1)}{(5s + 1)(s^2 + 6s + 25)}, \tag{50}$$

which is drawn from a well-known example [11].

From (50) it follows that $\omega_\ell = 0.2$, $\omega_{\mathrm{u}} = 5$.

The following experiments were carried out with ADAPLAB.

Experiment 1. Object (47) was excited by the test signal $u(t) = 0.03 \sin \omega_1 t$.

For $\omega_1 = \omega_1^{[1]} = 0.05$ and $\tau^{[1]} = 3750$, we obtained $\bar{T}_\ell(\omega_1^{[1]}, \tau^{[1]}) = 5.27$. For $\omega_1 = \omega_1^{[2]} = 0.0135$ and $\tau^{[2]} = 4800$, we obtained $\bar{T}_\ell(\omega_1^{[2]}, \tau^{[2]}) = 4.8$ and, consequently, the estimate of the lower bound is $\hat{\omega}_\ell^n = 0.208$.

Experiment 2. Object (47) was excited by the signal

$$u(t) = 1 \sin \omega_1 t.$$

For $\omega_1 = \omega_1^{[1]} = 20$ and $\tau^{[1]} = 6.28$, we obtained $\bar{T}_{\mathrm{u}}(\omega_1^{[1]}, \tau^{[1]}) = 0.095$. For $\omega_1 = \omega_1^{[2]} = 74$ and $\tau^{[2]} = 3.14$, we obtained $\bar{T}_{\mathrm{u}}(\omega_1^{[2]}, \tau^{[2]}) = 0.106$ and, consequently, $\hat{\omega}_{\mathrm{u}}^n = 9.4$.

8. CONCLUSIONS

A method for experimentally determining the bounds of eigenfrequencies of an object with unknown coefficients is elaborated. These bounds are assumed to be determined by the least and largest time constants in the denominator in its transfer function. These time constants strongly differ from other time constants (condition (19)).

Knowledge of the eigenfrequency bounds is helpful in widening the field of application of the identification algorithm 1. Therefore, our method, along with algorithm 1 and verification of the "small disturbance" condition (14), is an effective practical tool for identifying objects under unknown bounded disturbance.

APPENDIX

Proof of Assertion 3. Using the transfer function (15), we find that frequency parameters (3) are given by the expression

$$\alpha_k = \frac{\sum\limits_{q=0}^{r_2} \ell_{2q}\omega_k^{2q}}{\sum\limits_{q=0}^{n} m_{2q}\omega_k^{2q}}, \quad \beta_k = \frac{\sum\limits_{q=0}^{r_1} \ell_{2q+1}\omega_k^{2q+1}}{\sum\limits_{q=0}^{n} m_{2q}\omega_k^{2q}} \quad \left(k = \overline{1, n}\right), \tag{A.1}$$

where $\ell_0 = k_0,\ \ell_1 = k_1 - k_0 d_1,\ \ell_2 = -k_2 d_0 + k_1 d_1 - k_0 d_2, \ldots$. The coefficients $k_0, k_1, d_0, \ldots$ are related to time constants by the expression

$$k_1 = \left(\sum_{i=1}^{p_3} \check{T}_1 + \sum_{i=1}^{p_4} 2\, \tilde{\check{T}}_i \tilde{\check{\xi}}_i \right) k, \quad d_1 = \sum_{i=1}^{p_1} \bar{T}_i + \sum_{i=1}^{p_2} 2\, \tilde{\bar{T}}_i \tilde{\bar{\xi}}_i \quad k_0 = k. \tag{A.2}$$

Let

$$\alpha_1 = \alpha_1^\ell + \varepsilon_\alpha^\ell(\omega_1), \quad \beta_1 = \beta_1^\ell + \varepsilon_\beta^\ell(\omega_1), \tag{A.3}$$

where

$$\alpha_1^\ell = \ell_0, \quad \beta_1^\ell = \ell_1 \omega_1. \tag{A.4}$$

Using (A.1), since $m_0 = 1$, the functions $\varepsilon_\alpha^\ell(\omega_1)$ and $\varepsilon_\beta^\ell(\omega_1)$ can be expressed as

$$\varepsilon_\beta^\ell(\omega_1) = \omega_1^3 \bar{\varepsilon}_\beta^\ell(\omega_1), \quad \varepsilon_\alpha^\ell(\omega_1) = \omega_1^2 \bar{\varepsilon}_\alpha^\ell(\omega_1), \tag{A.5}$$

where the functions $\bar{\varepsilon}_\alpha^\ell(\omega_1)$ and $\bar{\varepsilon}_\beta^\ell(\omega_1)$ are bounded for $\omega_1 < \omega_\ell$.

Using the notation $\delta_\alpha(\tau) = (\hat{\alpha}_1 - \alpha_1)/\alpha_1$ and $\delta_\beta(\tau) = (\hat{\beta}_1 - \beta_1)/\beta_1$, we can represent the estimates of frequency parameters as

$$\begin{aligned} \hat{\alpha}_1 &= \alpha_1[1 + \delta_\alpha(\tau)] = \left[\alpha_1^\ell + \varepsilon_\alpha^\ell(\omega_1)\right][1 + \delta_\alpha(\tau)], \\ \hat{\beta}_1 &= \beta_1[1 + \delta_\beta(\tau)] = \left[\beta_1^\ell + \varepsilon_\beta^\ell(\omega_1)\right][1 + \delta_\beta(\tau)]. \end{aligned} \tag{A.6}$$

Substituting these expressions into formula (28), we obtain a relation of the type (30)

$$\bar{T}_1(\omega_1, \tau) = -\frac{\beta_1^\ell}{\omega_1 \alpha_1^\ell} + \varepsilon_\ell(\omega_1, \tau), \tag{A.7}$$

in which

$$\varepsilon_\ell(\omega_1, \tau) = -\frac{\beta_1^\ell \delta_\beta(\tau) + \varepsilon_\beta^\ell(\omega_1)[1 + \delta_\beta(\tau)]}{\omega_1[\alpha_1^\ell + \varepsilon_\alpha^\ell(\omega_1)][1 + \delta_\alpha(\tau)]} + \frac{\beta_1^\ell}{\alpha_1^\ell} \frac{\{\alpha_1^\ell \delta_\alpha(\tau) + \varepsilon_\alpha^\ell(\omega_1)[1 + \delta_\alpha(\tau)]\}}{\omega_1[\alpha_1^\ell + \varepsilon_\alpha^\ell(\omega_1)][1 + \delta_\alpha(\tau)]}. \tag{A.8}$$

Using (A.4), let us express the first term of the sum in (A.7) as

$$T_\ell = -\frac{\ell_1}{\ell_0} = -\frac{k_1 - k_0 d_1}{k_0} = d_1 - \frac{k_1}{k_0}. \tag{A.9}$$

Hence, by virtue of (A.2), we obtain formula (23).

Using (9), (10), and (A.5), we can derive inequality (31) from (A.8).

Proof of Assertion 4.

Let us express the frequency parameters as

$$\begin{aligned} \hat{\alpha}_1 &= \alpha_1[1 + \delta_\alpha(\tau)] = [\alpha_1^{\mathrm{u}} + \varepsilon_\alpha^{\mathrm{u}}(\omega_1)][1 + \delta_\alpha(\tau)], \\ \hat{\beta}_1 &= \beta_1[1 + \delta_\beta(\tau)] = \left[\beta_1^{\mathrm{u}} + \varepsilon_\beta^{\mathrm{u}}(\omega_1)\right][1 + \delta_\beta(\tau)], \end{aligned} \tag{A.10}$$

where $\alpha_1^{\mathrm{u}} = \dfrac{\ell_{2r_2} \omega_1^{2r_2}}{m(\omega_1)}$, $\varepsilon_\alpha^{\mathrm{u}}(\omega_1) = \dfrac{\sum_{q=0}^{r_2-1} \ell_{2q} \omega_1^{2q}}{m(\omega_1)}$, $\beta_1^{\mathrm{u}} = \dfrac{\ell_{2r_1+1} \omega_1^{2r_1+1}}{m(\omega_1)}$, $\varepsilon_\beta^{\mathrm{u}}(\omega_1) = \dfrac{\sum_{q=0}^{r_1-1} \ell_{2q+1} \omega_1^{2q+1}}{m(\omega_1)}$, and

$$m(\omega_1) = \sum_{q=0}^{n} m_{2q} \omega_1^{2q}, \quad r_2 = r_1 + 1. \tag{A.11}$$

Substituting (A.10) and (A.11) into formula (36), we obtain the first structure in (38), where

$$(\omega_{\mathrm{u}}^{n})^{-1} = \frac{\alpha_1^{\mathrm{u}}}{\omega_1 \beta_1^{\mathrm{u}}} = \frac{\ell_{2r_2}}{\ell_{2r_1+1}} = \frac{k_\gamma d_n}{-k_{\gamma-1} d_n + k_\gamma d_{n-1}}.$$

Using the relation of the coefficients d_n, d_{n-1}, k_γ, and $k_{\gamma-1}$ with time constants, we obtain (22). Inequality (39) is demonstrated along the same lines as (31), because the approximation error is such that $\varepsilon_\alpha^{\mathrm{u}} = \omega_1^{-2} \bar{\varepsilon}_\alpha^{\mathrm{u}}(\omega_1)$ and $\varepsilon_\beta^{\mathrm{u}} = \omega_1^{-2} \bar{\varepsilon}_\beta^{\mathrm{u}}(\omega_1)$, where the functions $\bar{\varepsilon}_\alpha^{\mathrm{u}}(\omega_1)$ and $\bar{\varepsilon}_\beta^{\mathrm{u}}(\omega_1)$ are bounded for $\omega_1 > \omega_{\mathrm{u}}$.

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