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# A Frequency Adaptive Control for Multidimensional Systems

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**Abstract**—An adaptive control is designed for a multidimensional system with unknown constant coefficients under bounded polyharmonic disturbances containing an infinite number of harmonics of unknown amplitudes and frequencies. It uses a very small test signal. The control aim is to ensure given bounds for the forced oscillations in the output of the system and controller. Adaptation is based on finite-frequency identification of the system and a closed-loop system. By way of example, an adaptive control of a real physical system is given.

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## 1. INTRODUCTION

There are several trends in adaptive control under unknown bounded external disturbances.

The first of them is related to reference-model systems. Adaptive controls for such systems were first designed without regard for external disturbances [1–3]. Subsequently these systems, as shown in [4], were found to lose stability under the action of external disturbances. Thus there emerged a large number of papers concerned with the design of adaptive control algorithms for stabilization in these and other tracking systems under external disturbances. Typical results of this trend are outlined in [5, 6]. This approach can be illustrated with the example of [5], where an $LQ$-optimization problem is solved for a system with unknown coefficients [5]. To solve a problem expressed as Riccati equations, the true coefficients of the system are replaced by their quasi-estimates found by the gradient method. They may considerably differ from true coefficients since the identification problem has no solution under unknown external disturbances (if the test signals described below are neglected). Therefore, quasi-estimates are possible values for the coefficients consistent with the input and output of the system. Adaptation process is shown to converge to some unknown tracking error. Other adaptation methods without the use of quasi-estimates are described in [7].

The method of recurrent aim inequalities [8, 9] laid the foundation for the second trend. In this trend, adaptive control aim is expressed as constraints (margins) for the deviation of the steady output of the system. The solution of the $l_1$-optimization problem [10, 11] is extended in [12, 13] to a system with unknown coefficients. In these papers, quasi-estimates are determined by a special gradient method for the deviation of the steady-state output to be minimal. It is not easy to realize the adaptive control algorithm numerically. This is the cost to be paid for the best adjustment accuracy it provides under unknown coefficients and arbitrary bounded external disturbances.

Therefore, many papers restrict external disturbances to a narrower class. An unknown constant disturbance is used in [14]. The adaptive control algorithm for this case is simple in realization. In [15], external disturbance is defined by a piecewise-constant bounded function with a known frequency range. Adaption aim is a given characteristic polynomial of a closed-loop system (which is also used in [16, 17], where the external disturbance is an arbitrary bounded function). Coefficients of the system are estimated with an adaptive observer, and the control law (which is formed from these estimates and state vector estimate) contains a test signal.

In frequency adaptive control [18], as in the second trend, control aim is the magnitude of the steady-state output of a system. External disturbance is the sum of an infinite number of harmonics of unknown amplitudes and frequencies and sum of amplitudes bounded by a known number. The system and a closed-loop system are identified by the finite-frequency identification method [19], in which a system or a closed-loop system is excited by a test signal—sum of harmonics, whose number is not greater than the dimension of the state space of the system or the closed-loop system. The frequency of the test signal must not be the same as that of the external disturbance. This condition somewhat narrows the class of external disturbances, and is verified during identification.

In the adaptation methods described above, the controller is continuously adjusted, whereas controller parameters are adjusted after large time intervals (adaptation intervals) in frequency adaptive control for guaranteeing the linearity of the model of the system on these intervals (while in other methods, the model is nonlinear and conditions cannot be easily found such that the input and output do not take unduly large values during adaptation). Therefore, the adaptation algorithm can be numerically realized without any serious difficulties [20].

In this paper,[1] results of [18] are extended to multidimensional systems. Here we encounter two difficulties. This first is the determination of a relation between the steady values of adjusted variables and weight coefficients of the $H_\infty$-norm of the transfer matrix of the closed-loop system. The second is the determination of adaptation termination conditions. Termination is obviously implemented by comparing the matrices describing the system at the current and preceding adaptation intervals. For this purpose, these matrices must be uniquely represented. Such a comparison is possible only if matrices are represented in canonical form, which in this paper is taken to be the Luenberger column observable canonical form [22].

In Section 2, we formulate the adaptive control design problem and solve it in Section 3 for a system with known coefficients. Sections 4 and 5 are devoted to identification of a system (independently and in a closed-loop system, respectively). In Section 6, we state conditions for the adaptation process to converge. In Section 7, we describe an adaptive control for a gyroplatform.

## 2. FORMULATION OF THE PROBLEM

Let us consider the linear stationary system described by the equations

$$\dot{\mathbf{x}}_p = A_p\mathbf{x}_p + B_p(\mathbf{u}+\mathbf{f}), \quad \mathbf{y} = \mathbf{z} = C_p\mathbf{x}_p, \quad t \geq t_0, \tag{1}$$

$$\dot{\mathbf{x}}_c = A_c\mathbf{x}_c + B_c\mathbf{y}, \quad \mathbf{u} = C_c\mathbf{x}_c, \tag{2}$$

where $\mathbf{x}_p(t) \in R^n$ is the state vector of system (1), $\mathbf{x}_c(t) \in R^n$ is the state vector of controller (2), $\mathbf{u}(t) \in R^m$ is the control vector, $\mathbf{y}(t) \in R^r$ is a vector of measured variables, $\mathbf{z}(t) \in R^r$ is a vector of adjusted variables, $\mathbf{f}(t) \in R^m$ is a vector of unmeasurable external disturbances—bounded polyharmonic functions

$$f_j(t) = \sum_{k=1}^{\infty} f_{jk} \sin\left(\omega_k^f t + \varphi_{jk}\right), \quad j = \overline{1,m}, \tag{3}$$

[1] This paper is a revised version of report [21] read at the 15th IFAC Congress in Barcelona.

whose frequencies $\omega_k^f$ and phases $\varphi_{jk}$ ($j = \overline{1,m},\ k = \overline{1,\infty}$) are unknown, and amplitudes $f_{jk}$ satisfy the conditions

$$\sum_{k=1}^{\infty} f_{jk}^2 \le f_j^{*2}, \quad j = \overline{1,m}, \tag{4}$$

in which $f_j^*(j = \overline{1,m})$ are given numbers. Here $A_p$, $B_p$, $C_p$, $A_c$, $B_c$, and $C_c$ are number matrices. The pair $(A_p, B_p)$ is assumed to be controllable and the pair $(A_p, C_p)$, observable.

Forced oscillations at the outputs of the system and controller as $t \to \infty$ are

$$z_i(t) = \sum_{k=1}^{\infty} \overline{z}_i(\omega_k^f) \sin\left[\omega_k^f t + \varphi_i^z(\omega_k^f)\right], \quad i = \overline{1,r},$$
$$u_j(t) = \sum_{k=1}^{\infty} \overline{u}_j(\omega_k^f) \sin\left[\omega_k^f t + \varphi_j^u(\omega_k^f)\right], \quad j = \overline{1,m}.$$

The matrices $A_p$, $B_p$, and $C_p$ of system (1) are such that there exist matrices $A_c$, $B_c$, and $C_c$ of controller (2) for which the amplitudes of forced oscillations of the outputs of the system and controller satisfy the conditions

$$\sum_{k=1}^{\infty} \overline{z}_i^2(\omega_k^f) \le \overline{z}_i^{*2}, \quad i = \overline{1,r} \quad \text{and} \quad \sum_{k=1}^{\infty} \overline{u}_j^2(\omega_k^f) \le \overline{u}_j^{*2}, \quad j = \overline{1,m}, \tag{5}$$

where $\overline{z}_i^*$ and $\overline{u}_j^*$ ($i = \overline{1,r},\ j = \overline{1,m}$) are given numbers.

Let the matrices $A_p$, $B_p$, and $C_p$ be not known. To design controller (2), let us apply an adaptive control described by equations with piecewise-constant coefficients

$$\dot{\mathbf{x}}_c^{(\kappa)} = A_c^{(\kappa)}\mathbf{x}_c^{(\kappa)} + B_c^{(\kappa)}\mathbf{y} + L\mathbf{v}^{(\kappa)}, \quad \mathbf{u} = C_c^{(\kappa)}\mathbf{x}_c^{(\kappa)}, \quad t_{\kappa-1} \le t < t_\kappa, \quad \kappa = \overline{1,N}. \tag{6}$$

In these equations, $\kappa$ ($\kappa = \overline{1,N}$) is the number of the adaptation interval, $t_\kappa$ is the instant of completion of the $\kappa$th interval. The instant $t_\kappa$, like the number $N$ and matrices $A_c^{(\kappa)}$, $B_c^{(\kappa)}$, and $C_c^{(\kappa)}$, are determined during adaptation, $L$ is a given matrix, and $\mathbf{v}^{(\kappa)}(t) \in R^m$ is a vector of test actions, whose components are defined below.

Upon completion of adaptation, the controller at instant $t_N$ is described by Eqs. (2), in which $A_c = A_c^{(N)}$, $B_c = B_c^{(N)}$, and $C_c = C_c^{(N)}$.

**Problem 1.** *Find an adaptation algorithm for the coefficients of controller* (6) *such that system* (1), (2) *satisfies conditions* (5) *for the steady-state amplitudes of forced oscillations.*

## 3. CONTROL FOR A KNOWN SYSTEM

If the matrices $A_p$, $B_p$, and $C_p$ of system (1) are known, then the matrices of controller (2) guaranteeing conditions (5) are defined by expressions [23]

$$\begin{aligned} A_c &= A_p - B_p(R^{-1} - \gamma^{-2}Q_1)B_p^{\mathrm{T}}P - K_f C_p, \quad B_c = K_f, \\ C_c &= -R^{-1}B_p^{\mathrm{T}}P, \quad K_f = (E_n - \gamma^{-2}YP)^{-1}YC_p^{\mathrm{T}}, \end{aligned} \tag{7}$$

in which the $n \times n$ nonnegative matrices $P$ and $Y$ are the solutions of the Riccati equations

$$A_p^{\mathrm{T}}P + PA_p - PB_p(R^{-1} - \gamma^{-2}Q_1)B_p^{\mathrm{T}}P = -C_p^{\mathrm{T}}QC_p, \tag{8}$$
$$A_pY + YA_p^{\mathrm{T}} - YC_p^{\mathrm{T}}(E_r - \gamma^{-2}Q)C_pY = -B_pQ_1B_p^{\mathrm{T}}, \tag{9}$$

with number $\gamma$ satisfying the condition

$$\lambda_{\max}(PY) < \gamma^2, \tag{10}$$

where $\lambda_{\max}(M)$ is the largest eigenvalue of the nonnegative matrix $M$.

**Remark 1.** For $Q = E_r$ and $R = Q_1 = E_m$ ($E_n$ is an $n \times n$ unit matrix), Eqs. (8) and (9) coincide with the equations of the $H_\infty$-suboptimal control [24] (for $B_1 = B_2 = B_p$ and $C_1 = C_2 = C_p$).

Let $Q = \operatorname{diag}(q_1, q_2, \ldots, q_r)$, $R = \operatorname{diag}(r_1, r_2, \ldots, r_m)$, and $Q_1 = E_m$.

**Assertion 1.** *If the elements of the matrices $Q$ and $R$ satisfy the inequalities*

$$q_i \geq \frac{1}{\overline{z}_i^{*2}} \sum_{k=1}^{m} f_k^{*2}, \quad i = \overline{1, r} \quad \textit{and} \quad r_j \geq \frac{1}{\overline{u}_j^{*2}} \sum_{k=1}^{m} f_k^{*2}, \quad j = \overline{1, m}, \tag{11}$$

*then the steady-state amplitudes of forced oscillations of system* (1), (2) *with matrices* (7)–(9) *satisfy the inequality*

$$\sum_{i=1}^{r} \frac{1}{\overline{z}_i^{*2}} \sum_{k=1}^{\infty} \overline{z}_i^2(\omega_k^f) + \sum_{j=1}^{m} \frac{1}{\overline{u}_j^{*2}} \sum_{k=1}^{\infty} \overline{u}_j^2(\omega_k^f) \leq \gamma^{*2}, \tag{12}$$

*in which $\gamma^*$ is the least $\gamma$ for which $P$ and $Y$ are nonnegative matrices and condition* (10) *is satisfied.*

The proof Assertion 1 is given in the Appendix.

Inequality (12), in turn, implies that controller (2) with coefficients (7) guarantees conditions (5) imposed on the amplitudes of oscillations if $\gamma^* \leq 1$.

## 4. THE FIRST ADAPTATION INTERVAL

### *4.1. Frequency Parameters of a System*

For the sake of simplicity of presentation, we assume that system (1) is asymptotically stable. To estimate its description matrices, let us apply at the last matrix input $m$ test vectors

$$\mathbf{u}_j(t) = \mathbf{e}_j \sum_{k=1}^{n} \rho_{jk}^u \sin \omega_k (t - t_0), \quad t_0 + (j-1)\tau^{(1)} \leq t < t_0 + j\tau^{(1)}, \quad j = \overline{1, m}, \tag{13}$$

where $\rho_{jk}^u$ is the amplitude of the $k$th harmonic of the test action of the $j$th experiment and $\omega_k$ is the frequency of the $k$th harmonic (for which $\rho_{jk}^u > 0$, $\omega_k \neq 0$ ($j = \overline{1, m}$, $k = \overline{1, n}$) and $|\omega_i| \neq |\omega_j|$ ($i \neq j$)), $\mathbf{e}_j = \operatorname{col}_j E_m$ is the $j$th column of the matrix $E_m$, and $\tau^{(1)}$ is the duration of the $j$th experiment, i.e., a given number for which $t_0 + m\tau^{(1)} = t_1$ (it can be found experimentally from the necessary conditions [21] for the identification process to converge).

Let us apply the outputs $\mathbf{y}_j(t)$ ($j = \overline{1, m}$) of the system to the inputs of a Fourier filter, whose outputs are the estimates

$$\begin{gathered}
\widehat{\phi}_{ijk} = \phi_{ijk}\left(\tau^{(1)}\right) = \frac{2}{\rho_{jk}^u \tau^{(1)}} \int\limits_{t_0 + (j-1)\tau^{(1)}}^{t_0 + j\tau^{(1)}} y_{ij}(t) \sin \omega_k (t - t_0)\, dt, \\
\widehat{\psi}_{ijk} = \psi_{ijk}\left(\tau^{(1)}\right) = \frac{2}{\rho_{jk}^u \tau^{(1)}} \int\limits_{t_0 + (j-1)\tau^{(1)}}^{t_0 + j\tau^{(1)}} y_{ij}(t) \cos \omega_k (t - t_0)\, dt, \\
i = \overline{1, r}, \quad j = \overline{1, m}, \quad k = \overline{1, n},
\end{gathered} \tag{14}$$

of the elements $\phi_{ijk}$ and $\psi_{ijk}$ of the matrices $\Phi_k = \operatorname{Re} W(j\omega_k)$ and $\Psi_k = \operatorname{Im} W(j\omega_k)$ ($k = \overline{1, n}$) *of frequency parameters* [25] *of system* (1), where $W(s) = C_p(E_n s - A_p)^{-1} B_p$ is its transfer matrix and $y_{ij}(t)$ is the $i$th component of the vector $\mathbf{y}_j(t)$ found in the $j$th experiment.

*4.2. Identification of a System*

System (1) is identified as a canonical Luenberger form [22]

$$\dot{\mathbf{x}} = A\mathbf{x} + B(\mathbf{u} + \mathbf{f}), \quad \mathbf{y} = \mathbf{z} = C\mathbf{x}, \quad t \geq t_0. \tag{15}$$

The blocks $A_{ij}$ and $\mathbf{c}_{ij}$ ($i = \overline{1,r},\ j = \overline{1,r}$) of its matrices $A$ and $C$ are of special structure

$$A_{ii} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_{ii}^{[0]} \\ 1 & 0 & \cdots & 0 & -a_{ii}^{[1]} \\ 0 & 1 & \cdots & 0 & -a_{ii}^{[2]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{ii}^{[\nu_i-1]} \end{pmatrix}, \quad A_{i\neq j} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_{ij}^{[0]} \\ 0 & 0 & \cdots & 0 & -a_{ij}^{[1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{ij}^{[\nu_{ij}-1]} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}; \tag{16}$$

$$\mathbf{c}_{ii} = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \mathbf{c}_{i>j} = \begin{pmatrix} 0 & \cdots & 0 & -c_{ij} \end{pmatrix}, \quad \mathbf{c}_{i<j} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix},$$

where $\nu_{ij} = \min(\nu_i,\, \nu_j)$ and $\nu_i$ ($i = \overline{1,r}$) are the *observability indexes* [22] *of the system*, which are taken to be known for simplicity. The matrix $B$ consists of blocks $\mathbf{b}_{ij} = \mathrm{col}\left(b_{ij}^{[0]},\, b_{ij}^{[1]},\, \ldots,\, b_{ij}^{[\nu_i-1]}\right)$ ($i = \overline{1,r},\ j = \overline{1,m}$).

The coefficients of the matrices $A^{(1)} = \widehat{A}$ and $C^{(1)} = \widehat{C}$ are found from the equalities [26]

$$\begin{gathered} \widehat{c}_{ij} - \sum_{k=j+1}^{i-1} \widehat{c}_{ik}(\check{\nu}_{kj} - \nu_{kj})\widehat{g}_{kj}^{[\check{\nu}_{kj}-1]} + (\check{\nu}_{ij} - \nu_{ij})\widehat{g}_{ij}^{[\check{\nu}_{ij}-1]} = 0, \quad i = \overline{j+1,r}, \quad j = \overline{1,r-2}, \\ \widehat{a}_{ij}^{[k]} = \widehat{g}_{ij}^{[k]} - \sum_{l=j+1}^{r} \widehat{g}_{il}^{[k]}\, \widehat{c}_{lj}, \quad k = \overline{0,\nu_{ij}-1}, \quad i = \overline{1,r}, \quad j = \overline{1,r}, \end{gathered} \tag{17}$$

in which $\check{\nu}_{kk} = \nu_k$ and $\check{\nu}_{ki} = \min(\nu_k, \nu_i)$ for $k < i$, and $\check{\nu}_{ki} = \min(\nu_k + 1, \nu_i)$ for $k > i$, and $\widehat{g}_{il}^{[k]}$ ($k = \overline{0,\nu_{ij}-1},\ i = \overline{1,r},\ j = \overline{1,r}$). The coefficients of the matrix $B^{(1)} = \widehat{B}$ are determined from the system of frequency identification equations [26]

$$\sum_{i=1}^{m}\sum_{j=0}^{\nu_k-1} \mathit{\Omega}^j \mathbf{i}_i\, \widehat{b}_{ki}^{[j]} + \sum_{i=1}^{r}\sum_{j=0}^{\check{\nu}_{ki}-1} \mathit{\Omega}^j \widehat{\mathbf{h}}_i\, \widehat{g}_{ki}^{[j]} = -\mathit{\Omega}^{\nu_k}\widehat{\mathbf{h}}_k, \quad k = \overline{1,r}, \tag{18}$$

in which

$$\begin{gathered} \mathit{\Omega} = \mathrm{diag}\,(\,\omega_1,\, \omega_2,\, \ldots,\, \omega_n\,) \otimes J \otimes E_m, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{i}_i = \mathrm{col}_i\, I\ (i = \overline{1,m}), \quad \widehat{\mathbf{h}}_i = \mathrm{col}_i\, \widehat{H}\ (i = \overline{1,r}), \\ I = \begin{pmatrix} E_m & 0_m & E_m & 0_m & \ldots & E_m & 0_m \end{pmatrix}^{\mathrm{T}}, \\ \widehat{H} = \begin{pmatrix} -\widehat{\mathit{\Phi}}_1 & -\widehat{\mathit{\Psi}}_1 & -\widehat{\mathit{\Phi}}_2 & -\widehat{\mathit{\Psi}}_2 & \ldots & -\widehat{\mathit{\Phi}}_n & -\widehat{\mathit{\Psi}}_n \end{pmatrix}^{\mathrm{T}}, \end{gathered}$$

and $0_m$ is an $m \times m$ zero block.

**Remark 2.** The solution of system (18) gives the coefficients of an equivalent system of (1) in "input-output" form

$$\widehat{G}(s)\mathbf{y} = \widehat{B}(s)(\mathbf{u} + \mathbf{f}), \quad t \geq t_0,$$

and polynomials of its matrices have a special structure [29]

$$\widehat{g}_{ii}(s) = \widehat{g}_{ii}^{[0]} + \widehat{g}_{ii}^{[1]}s + \ldots + \widehat{g}_{ii}^{[\nu_i-1]}s^{\nu_i-1} + s^{\nu_i}, \quad i = \overline{1,r},$$
$$\widehat{g}_{i\neq j}(s) = \widehat{g}_{ij}^{[0]} + \widehat{g}_{ij}^{[1]}s + \ldots + \widehat{g}_{ij}^{[\breve{\nu}_{ij}-1]}s^{\breve{\nu}_{ij}-1}, \quad i = \overline{1,r}, \quad j = \overline{1,r},$$
$$\widehat{b}_{ij}(s) = \widehat{b}_{ij}^{[0]} + \widehat{b}_{ij}^{[1]}s + \ldots + \widehat{b}_{ij}^{[\nu_i]}s^{\nu_i}, \quad i = \overline{1,r}, \quad j = \overline{1,m}.$$

Using transformations (17), we can estimate the coefficients of matrices (16) from the coefficients of the special polynomial matrix $\widehat{G}(s)$. For $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_r$, the coefficients are $\widehat{a}_{ij}^{[k]} = \widehat{g}_{ij}^{[k]}$ ($k = \overline{0, \nu_{ij}-1}$, $i = \overline{1,r}$, $j = \overline{1,r}$) of matrices (16) and the estimates for the coefficients of matrices (16) of the Luenberger canonical form are obviously determined directly from the solution of system (18) of frequency identification equations.

**Assertion 2** ([27]). *The solution of the limit system* (18) *as* ($\tau^{[1]} \to \infty$) *exists and is unique.*

**Corollary 1.** *By the properties* [28] *of continuous dependence of the solution on the matrices of system* (18) *and its vectors of free coefficients, the following limiting equalities hold:*

$$\lim_{\tau^{[1]}\to\infty} a_{ij}^{[k]}(\tau^{[1]}) = a_{ij}^{[k]}, \quad k = \overline{0, \nu_{ij}-1}, \quad i = \overline{1,r}, \quad j = \overline{1,r},$$
$$\lim_{\tau^{[1]}\to\infty} b_{ij}^{[k]}(\tau^{[1]}) = b_{ij}^{[k]}, \quad k = \overline{0, \nu_i-1}, \quad i = \overline{1,r}, \quad j = \overline{1,m},$$
$$\lim_{\tau^{[1]}\to\infty} c_{ij}(\tau^{[1]}) = c_{ij}, \quad i = \overline{1,r}, \quad j = \overline{1,r}.$$

### *4.3. Controller Design*

Using identification results, let us derive the Riccati equations (8) and (9), determine the elements of the matrices $Q$ and $R$ from inequalities (11), $Q_1 = E_m$, and replace the matrices $A_p$, $B_p$, and $C_p$ by their estimates $A^{(1)}$, $B^{(1)}$, and $C^{(1)}$. Solving these equations for different $\gamma$, we find the number $\gamma^*$ and compute using expressions (7), the matrices $A_c^{(2)}$, $B_c^{(2)}$, and $C_c^{(2)}$ of controller (6) for the second interval

$$\dot{\mathbf{x}}_c^{(2)} = A_c^{(2)}\mathbf{x}_c^{(2)} + B_c^{(2)}\mathbf{y} + L\mathbf{v}^{(2)}, \quad \mathbf{u} = C_c^{(2)}\mathbf{x}_c^{(2)}. \tag{19}$$

It is easy to show that the matrices $A_c^{(2)}$, $B_c^{(2)}$, and $C_c^{(2)}$ of this controller are determined from the matrices $A^{(1)}$, $B^{(1)}$, and $C^{(1)}$ by relations (7) within to a similarity transformation.

## 5. THE SECOND ADAPTATION INTERVAL

### *5.1. Frequency Parameters of a Closed-Loop System*

Let us excite system (1), (19) with $m$ test vectors

$$\mathbf{v}_j^{(2)}(t) = \mathbf{e}_j \sum_{k=1}^{n} \rho_{jk}^{v} \sin\omega_k(t-t_1), \quad t_1 + (j-1)\tau^{(2)} \leq t < t_1 + j\tau^{(2)}, \quad j = \overline{1,m},$$

where $\rho_{jk}^{v} > 0$ ($j = \overline{1,m}$) are the amplitudes of test signals of the closed-loop system.

The duration of every experiment is

$$\tau^{(2)} = \tau^{(1)} + K, \tag{20}$$

where $K$ is a given positive number and $t_2 = t_1 + m\tau^{(2)}$.

Applying the outputs $\mathbf{y}_j(t)$ $(j = \overline{1, m})$ of the closed-loop system (1) with controller (19) to the inputs of a Fourier filter, we obtain the estimates

$$\begin{aligned} \widehat{\theta}_{ijk} &= \frac{2}{\rho^v_{jk}\tau^{(2)}} \int\limits_{t_1+(j-1)\tau^{(2)}}^{t_1+j\tau^{(2)}} y_{ij}(t)\sin\omega_k(t-t_1)\,dt, \\ \widehat{\xi}_{ijk} &= \frac{2}{\rho^v_{jk}\tau^{(2)}} \int\limits_{t_1+(j-1)\tau^{(2)}}^{t_1+j\tau^{(2)}} y_{ij}(t)\cos\omega_k(t-t_1)\,dt, \end{aligned} \qquad i=\overline{1,r},\ j=\overline{1,m},\ k=\overline{1,n}, \tag{21}$$

for the elements $\theta_{ijk}$ and $\xi_{ijk}$ of the matrices $\Theta_k = \operatorname{Re} W_s(j\omega_k)$ and $\Xi_k = \operatorname{Im} W_s(j\omega_k)$ $(k = \overline{1, n})$ of *frequency parameters of the closed-loop system*

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_c^{(2)} \end{bmatrix} &= \begin{pmatrix} A & BC_c^{(2)} \\ B_c^{(2)}C & A_c^{(2)} \end{pmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^{(2)} \end{bmatrix} + \begin{pmatrix} 0_{n,m} \\ L \end{pmatrix} \mathbf{v}^{(2)} + \begin{pmatrix} B \\ 0_{n,m} \end{pmatrix} \mathbf{f}, \\ \mathbf{y} &= \begin{pmatrix} C & 0_{r,n} \end{pmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^{(2)} \end{bmatrix}, \end{aligned} \tag{22}$$

whose transfer matrix is

$$W_s(s) = [E_r - W(s)W_c(s)]^{-1} W(s) W_v(s), \tag{23}$$

where $W_c(s) = C_c^{(2)} \left(E_n s - A_c^{(2)}\right)^{-1} B_c^{(2)}$ and $W_v(s) = C_c^{(2)} \left(E_n s - A_c^{(2)}\right)^{-1} L$.

**Remark 3.** If the closed-loop system (22) becomes unstable, controller (19) must be disconnected and the input of system (13) must be formed in the third adaptation interval, increasing the filtration time

$$\tau^{(3)} = \tau^{(2)} + K \tag{24}$$

compared to that of the first interval. Then, solving the system of frequency identification Eqs. (18), we can find the matrices $A^{(3)}$, $B^{(3)}$, and $C^{(3)}$, and then solving the Riccati equations we can find the matrices $A_c^{(4)}$, $B_c^{(4)}$, and $C_c^{(4)}$ of the new controller. Filtration time must be increased until the closed-loop system becomes asymptotically stable.

### *5.2. Identification of a System*

Using the matrices $\widehat{\Theta}_k$ and $\widehat{\Xi}_k$ of estimates of frequency parameters of the closed-loop system (22), let us find new values for the matrices $\widehat{\Phi}_k = \Phi_k(\tau^{(2)})$ and $\widehat{\Psi}_k = \Psi_k(\tau^{(2)})$ $(k = \overline{1, n})$ of estimates of frequency parameters of the system. For this purpose, we use the relation

$$\Phi_k + j\Psi_k = [\Theta_k + j\Xi_k] \left\{ W_c(j\omega_k)[\Theta_k + j\Xi_k] + W_v(j\omega_k) \right\}^{-1}, \quad k = \overline{1, n}, \tag{25}$$

which obviously follows from (23).

In (25), replacing the matrices $\Theta_k$ and $\Xi_k$ by their estimates, we obtain new matrices $\widehat{\Phi}_k$ and $\widehat{\Psi}_k$ $(k = \overline{1, n})$ of estimates of frequency parameters of the system. Using these new matrices, we form a new system of frequency identification Eqs. (18). Solving this system, after simple transformations (17) we obtain the matrices $A^{(2)}$, $B^{(2)}$ and $C^{(2)}$ for the identified model (in canonical form (15)) of system (1).

Let us verify whether the conditions

$$\begin{aligned} a_{ij}^{[k](1)} \div a_{ij}^{[k](2)} &\le \varepsilon_a, \quad k=\overline{0,\nu_{ij}-1}, \quad i=\overline{1,r}, \quad j=\overline{1,r}, \\ b_{ij}^{[k](1)} \div b_{ij}^{[k](2)} &\le \varepsilon_b, \quad k=\overline{0,\nu_i-1}, \quad i=\overline{1,r}, \quad j=\overline{1,m}, \\ c_{ij}^{(1)} \div c_{ij}^{(2)} &\le \varepsilon_c, \quad i=\overline{1,r}, \quad j=\overline{1,r}, \end{aligned} \tag{26}$$

hold or not (closeness of coefficients of the systems identified in the first and second adaptation intervals), where $\div$ denotes the ratio $a \div b = |a-b|/|b|$ for $b \neq 0$, or $a \div b = |a|$ for $b = 0$, and $\varepsilon_a$, $\varepsilon_b$, and $\varepsilon_c$ are given numbers.

If these conditions are satisfied, then the adaptation process terminates at $N = 2$, and unknown matrices of controller (2) take the form $A_c = A_c^{(2)}$, $B_c = B_c^{(2)}$, and $C_c = C_c^{(2)}$, respectively. Otherwise (if the system identified on the first adaptation interval is not accurate), we must design a new controller (for the third adaptation interval), etc.

## 6. CONVERGENCE OF THE ADAPTION PROCESS

Let us introduce experimental filterability functions [30]

$$\begin{aligned} \ell_{ik}^{s}(\tau_0,\tau) &= \frac{2}{\tau}\int_{\tau_0}^{\tau} \overline{y}_i(t)\sin\omega_k(t-\tau_0)\,dt, \\ \ell_{ik}^{c}(\tau_0,\tau) &= \frac{2}{\tau}\int_{\tau_0}^{\tau} \overline{y}_i(t)\cos\omega_k(t-\tau_0)\,dt, \end{aligned} \qquad i=\overline{1,r},\ j=\overline{1,m},\ k=\overline{1,n}, \tag{27}$$

which are the outputs of a Fourier filter, whose inputs are fed with the "natural" output $\overline{\mathbf{y}}(t) = \mathrm{col}(\overline{y}_1(t), \overline{y}_2(t), \ldots, \overline{y}_r(t))$ (for $\mathbf{u}(t) = \mathbf{0}$) of system (1). The parameter $\tau_0$ defines the start of the experiment (verification of disturbances for ff-filterability).

A disturbance $\mathbf{f}(t)$ is said to be strictly ff-filterable [30] if

$$\lim_{\tau\to\infty} \ell_{ik}^{s}(\tau_0,\tau) = \lim_{\tau\to\infty} \ell_{ik}^{c}(\tau_0,\tau) = 0, \quad i=\overline{1,r}, \quad k=\overline{1,n}. \tag{28}$$

**Remark 4.** Condition (28) holds if the test frequencies do not coincide with the frequencies of external disturbances ($|\omega_k| \neq |\omega_j^f|$ ($k=\overline{1,n}$, $j=\overline{0,\infty}$)). If it does not hold, then other test frequencies must be chosen until the condition (28) is satisfied with sufficient accuracy.

**Assertion 3.** *If a disturbance* $\mathbf{f}(t)$ *is strictly ff-filterable, then estimates* (14) *as* $\tau\to\infty$ *converge to the frequency parameters of system* (1)*:*

$$\lim_{\tau\to\infty} \phi_{ijk}(\tau) = \phi_{ijk} \ \textit{and} \ \lim_{\tau\to\infty} \psi_{ijk}(\tau) = \psi_{ijk}, \quad i=\overline{1,r},\ j=\overline{1,m},\ k=\overline{1,n}. \tag{29}$$

The proof of Assertion 3 is given in [31] for a SISO system and extended to multidimensional systems in [32], where the convergence rates (29) are estimated.

**Assertion 4.** *If a disturbance* $\mathbf{f}(t)$ *is strictly ff-filterable, then the adaptation process converges and guarantees conditions* (5).

**Proof.** The adaptation process converges (by Assertion 3 and 2 and Corollary 1) if the filtration time $\overline{\tau}$ takes a large value on some interval of adaptation. That a large $\overline{\tau}$ can be attained is implied by conditions (20) and (24), which show that the length of every succeeding interval is greater than the length of the preceding interval by a given magnitude $K$:

$$\tau^{(\kappa)} = \tau^{(\kappa-1)} + K, \quad \kappa = \overline{1, N}.$$

The "natural" outputs $\overline{y}_{ij}(t)$ $(i = \overline{1, r},\ j = \overline{1, m})$ of system (1) and of system (1), (6) are also used in choosing amplitudes for test signals from the "small excitation" condition

$$\overline{y}_{ij}(t) \div y_{ij}(t) \leq \overline{\varepsilon}, \quad i = \overline{1, r}, \quad j = \overline{1, m},$$

where $\overline{\varepsilon}$ is a given number defining the admissible deviation of "natural" outputs of a system and a closed-loop system from their outputs under test signals.

## 7. AN EXAMPLE

### *7.1. Model of a System*

Let us consider the gyroplatform [33] described by the equations

$$\begin{gathered} \mathcal{P}\ddot{\boldsymbol{\beta}} + \mathcal{P}\mathcal{S}\dot{\boldsymbol{\omega}} + \mathcal{H}\mathcal{C}\boldsymbol{\omega} + \mathcal{N}\dot{\boldsymbol{\beta}} = \mathbf{0}, \\ \mathcal{J}\dot{\boldsymbol{\omega}} - \mathcal{C}^{\mathrm{T}}\mathcal{H}\mathcal{S}\boldsymbol{\omega} - \left(\mathcal{C}^{\mathrm{T}}\mathcal{H} + \mathcal{S}^{\mathrm{T}}\mathcal{N}\right)\dot{\boldsymbol{\beta}} = \mathcal{Q}(\mathbf{u} + \mathbf{f}), \end{gathered} \tag{30}$$

where $\beta_1$ and $\beta_2$ are precession angles (measured rotation angles) of gyroscopes, $\omega_1$ and $\omega_2$ are the projections of absolute angular velocities of the platform on its axes, i.e., the variables

$$\dot{\alpha}_1 = \omega_1 \quad \text{and} \quad \dot{\alpha}_2 = \omega_2, \tag{31}$$

in which $\alpha_1$ and $\alpha_2$ are stabilization angles (adjusted variables), $u_1$ and $u_2$ are torques of stabilization (control) motors, $f_1$ and $f_2$ are external disturbances, and

$$\mathcal{P} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \ \mathcal{Q} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \ \mathcal{N} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}, \ \mathcal{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},$$
$$\mathcal{J} = \begin{pmatrix} j_x & 0 \\ 0 & j_y \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} -\sin\delta_1 & \cos\delta_1 \\ \cos\delta_2 & \sin\delta_2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} -\cos\delta_1 & -\sin\delta_1 \\ -\sin\delta_2 & \cos\delta_2 \end{pmatrix},$$
$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The parameters of the gyroplatform are

$$\begin{gathered} p_1 = p_2 = 10^{-5}\,\text{kg} \times \text{m}^2, \quad q_1 = q_2 = 10^{-5}, \quad n_1 = n_2 = 4 \times 10^{-3}\,\text{Nms}, \\ h_1 = h_2 = 10^{-2}\,\text{Nms}, \quad j_x = 10^{-3}\,\text{kg} \times \text{and}/\text{s}^2, \quad j_y = 2 \times 10^{-3}\,\text{kg} \times \text{m}/\text{s}^2, \\ \delta_1 = -20^\circ, \quad \delta_2 = 30^\circ. \end{gathered} \tag{32}$$

Typical external disturbances acting on the gyroplatform are step or harmonic disturbances. Eight of the latter [23] are

$$f_1(t) = \rho_1^f \sin\omega_1^f t + \rho_2^f \cos\omega_2^f t, \quad f_2(t) = \rho_2^f \sin\omega_1^f t + \rho_1^f \cos\omega_2^f t, \tag{33}$$

where $\rho_1^f = 410$ and $\rho_2^f = 565$ are amplitudes and $\omega_1^f = 5$ and $\omega_2^f = 7$ are rolling frequencies of the gyroplatform base.

The gyroplatform controller is

$$\dot{\mathbf{x}}_c = A_c^{(1)}\mathbf{x}_c + B_c^{(1)}\boldsymbol{\beta}, \quad \mathbf{u} = C_c^{(1)}\mathbf{x}_c. \tag{34}$$

Its coefficients are determined in [23] from the values of parameters (32), $\gamma = \infty$, matrices $Q = 8 \times 10^{12} E_2$ and $Q_1 = 10^{20} E_2$, and $R = E_2$ of weight coefficients of Eqs (8) and (9). For the sake of brevity, here we omit the values of the coefficients of the matrices $A_c^{(1)}$, $B_c^{(1)}$, and $C_c^{(1)}$.

This controller in steady-state guarantees an adjustment accuracy of

$$|\alpha_{1,\mathrm{st}}| \le 3 \times 10^{-4}, \quad |\alpha_{2,\mathrm{st}}| \le 3 \times 10^{-4}. \tag{35}$$

### *7.2. Formulation of the Problem*

Let the kinetic moment $h_1$ at some (unknown) instant $t_1$ take the value $h_1^{(2)} < h_1^{(1)}$ due to the failure of one of the gyromotors. This situation is called the second mode of operation of the gyroplatform (unlike in the first mode, in which kinetic moments are equal $h_1^{(1)} = h_2$).

**Problem 2.** *Find the instant $t_1$, identify the system (second operation mode), and find new controller coefficients (adjust the controller to the new value of $h_1^{(2)}$ of the kinetic moment $h_1$) under which the accuracy condition (35) is satisfied.*

### *7.3. A Solution Method*

For a gyroplatform, the adjusted variables $\alpha_1$ and $\alpha_2$ are not controllable. Hence system (30) is completely controllable, but "system" (30), (31) is not controllable completely. On the other hand, using the first subsystem of Eqs. (30) we find a relation for the steady-state values of $\alpha_1$ and $\alpha_2$ with $\beta_1$ and $\beta_2$. In particular, for step disturbances ($\omega_1 = \omega_2 = 0$ in (33)) this relation is [33]

$$\alpha_{1,\mathrm{st}} = \frac{b_{11}}{h_1}\beta_{1,\mathrm{st}} + \frac{b_{12}}{h_2}\beta_{2,\mathrm{st}}, \qquad \alpha_{2,\mathrm{st}} = \frac{b_{21}}{h_1}\beta_{1,\mathrm{st}} + \frac{b_{22}}{h_2}\beta_{2,\mathrm{st}}, \tag{36}$$

where "st" denotes the steady-state values of variables and $b_{ij}$ $(i, j = 1, 2)$ are numbers defined by parameters (32) of the gyroplatform. Using a similar relation for the general case, we can reformulate problem 2, replacing condition (35) by conditions for the variables $\beta_1$ and $\beta_2$

$$|\beta_{1,\mathrm{st}}| \le \beta^*_{1,\mathrm{st}}, \qquad |\beta_{2,\mathrm{st}}| \le \beta^*_{2,\mathrm{st}}, \tag{37}$$

where $\beta^*_{1,\mathrm{st}}$ and $\beta^*_{2,\mathrm{st}}$ are numbers defined by relations (36) and similar relations and bounds for the steady-state error in inequalities (35). Replacing the aim condition (35) by condition (37), we can design a controller using only (30).

Note that changes in the kinetic moment $h_1$ have virtually no influence on the steady-state error due to the variables $\beta_1$ and $\beta_2$ since the gyroplatform controller has a sufficiently large gain. On the other hand, by expression (36), the adjusted variables $\alpha_1$ and $\alpha_2$ largely depend on the values of kinetic moments.

To illustrate our approach to solve problem 2, let us consider the first relation in (36), assuming for the sake of simplicity that $b_{12} = 0$ (i.e., gyroplatform parameter is $\delta_1 = 0$). Then by the equality $\alpha_{1,\mathrm{st}} = (b_{11}/h_1)\beta_{1,\mathrm{st}}$, the error $\beta_{1,\mathrm{st}}$ must be halved if the kinetic moment $h_1$ is halved. To attain this result, controller gains must be increased, i.e., we need a new controller, which is synthesized after the instant $t_1$ (when the new kinetic moment $h_1 = h_1^{(2)}$ is known).

Furthermore, the gyroplatform is not asymptotically stable (its characteristic polynomial has two zero roots). Therefore, an open-loop system (30) without a controller cannot be identified using a test signal (13). Hence we shall estimate its parameters, using the results of identification of a closed-loop system.

*7.4. Solution of the Problem*

We solve the problem with MATLAB, using its extension ADAPLAB-M [20] for finite-frequency identification and frequency adaptive control.

The problem is solved in steps:

(1) Filtering (21) the closed-loop system (30), (32), (34), we obtain the matrices $\widehat{\Theta}_k$ and $\widehat{\Xi}_k$ of estimates of frequency parameters.

(2) Using formulas (25) (in which $\widehat{W}_c(j\omega_k)$ and $\widehat{W}_v(j\omega_k)$ are computed from the coefficients of the matrices $A_c^{(1)}$, $B_c^{(1)}$, and $C_c^{(1)}$ of controller (34)), we find the values of the matrices $\widehat{\Phi}_k$ and $\widehat{\Psi}_k$ $(k = \overline{1,n})$ of estimates of frequency parameters of the system.

(3) Solving the system of frequency identification Eqs. (18), we find estimates for the coefficients of the gyroplatform model

$$\dot{\mathbf{x}} = \widehat{A}^{(1)}\mathbf{x} + \widehat{B}^{(1)}(\mathbf{u} + \mathbf{f}), \quad \boldsymbol{\beta} = \widehat{C}^{(1)}\mathbf{x}, \quad t \geq t_1,$$

in the first mode (with $h_1^{(1)} = 10^{-2}$Nms).

(4) Changing the kinetic moment $h_1 := h_1^{(2)} = 5 \times 10^{-3}$ Nms, we find (from the solution of system (18) of frequency identification equations and equalities (17)) the matrices $\widehat{A}^{(2)}$, $\widehat{B}^{(2)}$, and $\widehat{C}^{(2)}$ of the gyroplatform model in the second mode.

(5) Comparing the matrices of first- and second-mode models, we find a change in the kinetic moment weakly changes matrix coefficients (controller gains exceed the coefficients of the gyroplatform model to such an extend that "power failure" can be hardly be detected). Moreover, numerical experiments show a change in the kinetic moment $h_1$ (on the interval $[0.001, 0.01]$) exerts considerable influence on two nonzero minimal (in modulus) roots ($s_3$ and $s_4$) of the characteristic polynomial of the system (forming a complex-conjugate pair from identification results). Therefore, the instant $t_1$ is determined from the product $\widehat{s}_3\widehat{s}_4$ of roots of the identified system (its large change shows the *commencement of the second mode*). In the figure below, the product $s_3(h_1)s_4(h_1)$ is shown as a function of $h_1$ in the range $[0.001, 0.01]$.

**Remark 5.** For comparison, let us state the values of roots of the matrix $A^{(1)}$ constructed from parameters (32): $s_3^{(1)} = -12.41$ and $s_4^{(1)} = -29.08$, and their product $s_3^{(1)}s_4^{(1)} = 361$. In the second mode (with $h_1 = h_1^{(2)}$), the roots of the matrix $A^{(2)}$ are $s_3^{(2)} = -10.95 + j5.56$ and $s_4^{(2)} = -10.95 - j5.56$ and their product is $s_3^{(2)}s_4^{(2)} = 150.8$.

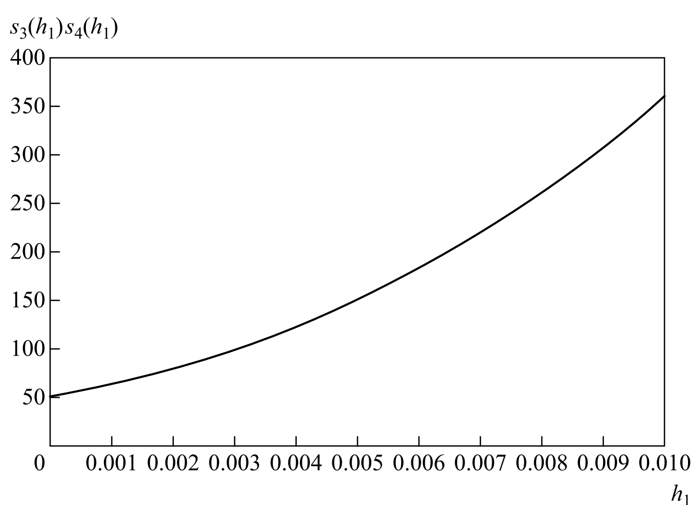


**Figure.**

Estimates for the roots of the matrix $\widehat{A}^{(1)}$ are

$$\widehat{s}_3^{(1)} = -12.48 \quad \text{and} \quad \widehat{s}_4^{(1)} = -31.74, \quad \text{and their product is} \quad \widehat{s}_3^{(1)}\widehat{s}_4^{(1)} = 396. \tag{38}$$

In the second mode (with $h_1^{(2)} = 5 \times 10^{-3}$Nms), estimates for these roots are

$$\widehat{s}_3^{(2)} = -11.96 + j5.51 \text{ and } \widehat{s}_4^{(2)} = -11.96 - j5.51, \text{ and their product is } \widehat{s}_3^{(2)}\widehat{s}_4^{(2)} = 173.3. \tag{39}$$

Hence a new controller must be designed.

(6) Controller

$$\dot{\mathbf{x}}_c = A_c^{(2)}\mathbf{x}_c + B_c^{(2)}\boldsymbol{\beta}, \quad \mathbf{u} = C_c^{(2)}\mathbf{x}_c \tag{40}$$

is designed by the $LQ$-procedure, which is the same as the $H_\infty$-suboptimal control procedure [23] if $\gamma \to \infty$. The coefficients of the matrix $Q_2$ were increased by

$$\left(\frac{\widehat{h}_1^{(1)}}{\widehat{h}_1^{(2)}}\right)^2 = \left(\frac{1.06 \times 10^{-2}}{5.71 \times 10^{-3}}\right)^2 = 3.45,$$

where $\widehat{h}_1^{(1)}$ for (38) and $\widehat{h}_1^{(2)}$ for (39) were determined from the curve shown in the figure.

The matrices of controller (40) take the values

$$A_c^{(2)} = \begin{pmatrix} -430.17 & -47\,575 & -4.1849 \times 10^{10} & -0.74481 & -20\,417 & 2.1756 \times 10^{10} \\ 1.5563 & 57.352 & -5.6250 \times 10^{7} & -0.39838 & -46.155 & -2.9828 \times 10^{7} \\ 0 & 1 & -1.0197 \times 10^{4} & 0 & 0 & -2.9460 \times 10^{3} \\ -311.89 & -37\,170 & -6.5962 \times 10^{10} & -256.45 & -61\,301 & -3.1553 \times 10^{9} \\ 0.98091 & 105.28 & -3.7611 \times 10^{7} & 0.69522 & -9.1289 & -8.4052 \times 10^{7} \\ 0 & 0 & -2.9646 \times 10^{3} & 0 & 1 & -1.2628 \times 10^{4} \end{pmatrix},$$

$$B_c^{(2)} = \begin{pmatrix} 4.1849 \times 10^{10} & -2.1757 \times 10^{10} \\ 5.6242 \times 10^{7} & 2.9827 \times 10^{7} \\ 9.7865 \times 10^{3} & 2.9452 \times 10^{3} \\ 6.5963 \times 10^{10} & 3.1527 \times 10^{9} \\ 3.7596 \times 10^{7} & 8.4050 \times 10^{7} \\ 2.9452 \times 10^{3} & 1.2236 \times 10^{4} \end{pmatrix},$$

$$C_c^{(2)} = \begin{pmatrix} -101.75 & -11\,095 & 43\,933 & 14.942 & -2\,082.6 & -49\,042 \\ -43.670 & -4\,039.7 & -79\,937 & 75.451 & 11\,650 & 524\,800 \end{pmatrix}.$$

(7) Modeling of the closed-loop system (30), (32), (40) shows that the new controller satisfies the accuracy conditions (35).

## 8. CONCLUSIONS

A new adaptive control for a multidimensional system under bounded polyharmonic disturbances (3) for guaranteeing adjustment accuracy conditions (5) is designed from experimentally determined frequency parameters of a system and a closed-loop system excited by a “sufficiently low” test signal.

## APPENDIX

**Proof of Assertion 1.** The following theorem [23] holds for the properties of transfer matrices $W_{zf}(s)$ and $W_{uf}(s)$ interconnecting the vectors of outputs $\mathbf{z}$ of a system and $\mathbf{u}$ of its controller with disturbance $\mathbf{f}$.

**Theorem 1.** *For the frequency matrix inequality*[2]

$$W_{zf}^{\mathrm{T}}(-j\omega^f)QW_{zf}(j\omega^f) + W_{uf}^{\mathrm{T}}(-j\omega^f)RW_{uf}(j\omega^f) \leq \gamma^2 E_m, \quad \omega^f \in [0,\infty), \tag{A.1}$$

*to hold, it is necessary and sufficient that the matrices of controller* (2) *be formed from relations* (7)–(9), *and condition* (10) *hold.*

Taking $\omega = \omega_k^f$ in (A.1) and multiplying it on the left by $\mathbf{f}_-^{[k]\mathrm{T}} = \mathrm{row}\,(f_{1k}e^{-j\varphi_{1k}},\, f_{2k}e^{-j\varphi_{2k}},\, \ldots,\, f_{mk}e^{-j\varphi_{mk}})$ and on the right by $\mathbf{f}_+^{[k]} = \mathrm{col}(\, f_{1k}e^{j\varphi_{1k}},\, f_{2k}e^{j\varphi_{2k}},\, \ldots,\, f_{mk}e^{j\varphi_{mk}}\,)$, we obtain

$$\mathbf{f}_-^{[k]\mathrm{T}} W_{zf}^{\mathrm{T}}(-j\omega_k^f)QW_{zf}(j\omega_k^f)\mathbf{f}_+^{[k]} + \mathbf{f}_-^{[k]\mathrm{T}} W_{uf}^{\mathrm{T}}(-j\omega_k^f)RW_{uf}(j\omega_k^f)\mathbf{f}_+^{[k]} \leq \gamma^2 \mathbf{f}_-^{[k]\mathrm{T}}\mathbf{f}_+^{[k]}. \tag{A.2}$$

It is easy to verify [23] that the amplitudes $\overline{z}_i(\omega_k^f)$ and $\overline{u}_j(\omega_k^f)$ of steady-state forced oscillations in each of coordinates of the vectors $\mathbf{z}$ and $\mathbf{u}$ are moduli of the elements of the complex-conjugate vectors

$$W_{zf}(j\omega_k^f)\mathbf{f}_+^{[k]} \text{ and } W_{zf}(-j\omega_k^f)\mathbf{f}_-^{[k]}, \quad \text{and} \quad W_{uf}(j\omega_k^f)\mathbf{f}_+^{[k]} \text{ and } W_{uf}(-j\omega_k^f)\mathbf{f}_-^{[k]},$$

which when replaced reduces (A.2) to the form

$$\sum_{i=1}^{r} q_i \overline{z}_i^2(\omega_k^f) + \sum_{j=1}^{m} r_j \overline{u}_j^2(\omega_k^f) \leq \gamma^2 \sum_{j=1}^{m} f_{jk}^2. \tag{A.3}$$

Adding inequalities (A.3) for all frequencies, by virtue of (4), we obtain

$$\sum_{i=1}^{r} q_i \sum_{k=1}^{\infty} \overline{z}_i^2(\omega_k^f) + \sum_{j=1}^{m} r_j \sum_{k=1}^{\infty} \overline{u}_j^2(\omega_k^f) \leq \gamma^2 \sum_{j=1}^{m}\sum_{k=1}^{\infty} f_{jk}^2 \leq \gamma^2 \sum_{j=1}^{m} f_j^{*2}.$$

This expression implies inequality (12) if the coefficients of the diagonal matrices $Q$ and $R$ satisfy conditions (11).

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[2] Here the sign $\leq$ is interpreted in the sense of sign definiteness of matrices: $A \leq B \sim \mathbf{x}^{\mathrm{T}}A\mathbf{x} \leq \mathbf{x}^{\mathrm{T}}B\mathbf{x}\ \forall\, \mathbf{x} \neq 0$.

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