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TOPICAL ISSUE

Controller Design in Precision and Speed. I. Minimal Phase One-dimensional Plants

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Abstract—A method to design the controllers of one-dimensional minimum phase plants under unknown bounded exogenous disturbances was proposed. It relies on determining the parameters of the Bézout identity providing the desired control precision and speed.

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1. INTRODUCTION

The automatic control systems are characterized by indices of which precision, speed, overshoot, and stability margins in phase and absolute value are the basic ones. The Bode plot method [1, 2] for the one-dimensional plants, that is, those with one measurable and one control variable, was the first method of design using these system indices.

Further extension of this method to the multidimensional plants proved to be onerous. The methods of LQ and H_∞ optimization [3–6] do not need the numbers of measured and control variables. Therefore, consideration was given to the indices of the optimal systems. In particular, their stability margins were established [7, 8], and the relation between the control precision and the structure and parameters of the optimization functionals was determined. On this basis, the methods of designing controllers of the multidimensional plants for the given precision and desired stability margins were developed [9].

Speed is an important index of the control systems. A method to design controllers in precision and speed for the minimum phase one-dimensional plants was suggested in [10]. It is based on solving the Bézout identity whose right side is the stable part of the polynomial for the extremals of a special functional providing system stability margins, precision, and speed.

The present paper proposes a method to generate the right side of the Bézout identity doing without determination of the roots of the aforementioned polynomial, which is especially important in the multidimensional case representing the subject matter of the second part of the present paper where the controller design from the requirements on precision, speed, and stability margins relies on the matrix Bézout identity. In distinction to [11, 10] where a similar problem is reduced to solving a nonlinear algebraic Riccati equation or to the problem of H_∞-optimization solved on the basis of the linear matrix inequalities, this identity is solved by attacking a system of linear algebraic equations.

The book of B.T. Polyak, M.V. Khlebnikov, and P.S. Shcherbakov [12] which determined a controller providing the given boundary of the quadratic form of the controlled variables is devoted to the methods of suppressing the exogenous disturbances. The coefficients of this quadratic form are the best in a sense. The method allows for the constraints on control and the impact of the initial conditions.

2. FORMULATION OF THE PROBLEM

2.1. Precision and Speed Indices

Let us consider an asymptotically stable control system obeying the equations

$$\begin{gathered} d_n y^{(n)} + d_{n-1} y^{(n-1)} + \ldots + d_1 \dot{y} + d_0 y \\ = k_m u^{(m)} + \ldots + k_1 \dot{u} + k_0 u + c_p f^{(p)} + \ldots + c_1 \dot{f} + c_0 f, \quad m < n, \quad p < n, \quad t_0 \leqslant t \leqslant t_1, \end{gathered} \tag{1}$$

$$g_{n_c} u^{(n_c)} + \ldots + g_1 \dot{u} + g_0 u = r_{m_c} y^{(m_c)} + \ldots + r_1 \dot{y} + r_0 y, \quad n_c \geqslant m_c, \tag{2}$$

where $y(t)$ is the measured output of plant (1) which is the controlled variable, $u(t)$ is the control generated by the controller (2), $f(t)$ is an unknown exogenous disturbance bounded by a certain number f^*. This system has zero initial conditions:

$$y(t_0) = \dot{y}(t_0) = \ldots = y^{(n-1)}(t_0) = 0, \quad u(t_0) = \dot{u}(t_0) = \ldots = u^{(n_c-1)}(t_0) = 0,$$

t_0 and t_1 are the given numbers.

To specify the form of the exogenous disturbance , we decompose the interval of system operation into N subintervals of duration $h = \frac{t_1 - t_0}{N}$ each and, using the fact of existence of the numbers $f(kh) = f(k)$, $k = \overline{1, N}$, expand $f(k)$ in the Fourier series

$$f(k) = \sum_{i=1}^{N} f_i \sin(\omega_i k + \varphi_{f,i}), \quad k = \overline{1, N},$$

where $f_i,\ i = \overline{1, N}$ are the coefficients of the Fourier series, $\omega_i = \frac{2\pi}{N} i,\ i = \overline{1, N}$. We assume now that the exogenous disturbance is a polyharmonic function

$$f(t) = \sum_{i=1}^{N} f_i \sin(\omega_i t + \varphi_{f,i}) \tag{3}$$

with unknown frequencies ω_i and phases $\varphi_{f,i},\ i = \overline{1, N}$, its unknown amplitudes satisfying the condition

$$\sum_{i=1}^{N} |f_i| \leqslant f^*. \tag{4}$$

The system output

$$y(t) = y_b(t) + y_{tr}(t)$$

consists of the *working* (basic) process $y_b(t)$ and the *transient* process $y_{tr}(t)$ directed toward the working one.

The working process is given by

$$y_b(t) = \sum_{i=0}^{N} a(\omega_i) \sin(\omega_i t + \varphi_i), \tag{5}$$

where $a(\omega_i)$ and $\varphi_i,\ i = \overline{1, N}$, are, respectively, the amplitudes and phases of the system output.

If the exogenous disturbance (3) consists of one or more actually existing harmonics, then (5) is called the stationary process. In the case at hand, such harmonics may miss (for example, $f(t)$ is a linear function or exponent), and the frequencies $\omega_i,\ i = \overline{1, N}$, are the result of expansion into the Fourier series of the almost exogenous disturbance. These frequencies depend on the choice of lengths of the intervals $[t_0, t_1]$ and h. Therefore, we use the notion of the working process.

We introduce the notions of indices of system (1), (2) which generalize the well known indices for the standard reference signals [1, 2].

Control precision is the least positive number y_b^a such that the control error satisfies the condition

$$|y_b(t)| \leqslant y_b^a, \quad t_0 \leqslant t \leqslant t_1.$$

The system output can exceed the value y_b^a

$$\sup_{t_0 \leqslant t \leqslant t_1} |y(t)| > y_b^a$$

because of the transient process generated by the initial conditions and the initial value of the exogenous disturbance.

The duration and relative value of exceeding the number y_b^a are characterized by the control time and overshoot.

Control time is the least time t_{reg} when the inequality

$$|y(t) - y_b(t)| \leqslant \varepsilon, \quad t \geqslant t_{\text{reg}}$$

is satisfied, where ε is the given positive number.

The value $\varepsilon = 0.05 \sup_{t_0 \leqslant t \leqslant t_1} |y(t)|$ is accepted for the unit step-type or harmonic reference signal [1, 2].

Overshoot is given by

$$\sigma = \frac{\sup\limits_{t_0 \leqslant t \leqslant t_1} |y(t)| - \sup\limits_{t_0 \leqslant t \leqslant t_1} |y_b(t)|}{\sup\limits_{t_0 \leqslant t \leqslant t_1} |y_b(t)|} \times 100\%.$$

The phase stability margins (φ_{z}) and absolute value (L) [1, 2] of system (1), (2) are established by applying to plant (1) the signal $(-\sin \omega t)$ instead of the control and measuring the controller output $u = a(\omega)\sin(\omega t + \varphi(\omega))$. The stability margins are determined using the functions $a(\omega)$ and $\varphi(\omega)$.

With such definition, the system may lose stability because of its opening. To determine the stability margin, one makes use, therefore, of its radius [9]

$$r_a = \inf_{0 \leqslant \omega < \infty} |\mathrm{v}(j\omega)|, \tag{6}$$

where $\mathrm{v}(j\omega) = 1 + a(\omega)e^{j\varphi(\omega)} = 1 + w(j\omega)$ is the function of recurrent difference with the transfer function of the open-loop system $w(j\omega)$. It is determined experimentally without opening the system. If $r_a \geqslant 0.75$, then $\varphi_{\text{z}} \geqslant 42°$, $L \geqslant 1.7$, and $\varphi_{\text{z}} \geqslant 60°$, $L \geqslant 2.0$ for $r_a \geqslant 1$.

Problem 2.1 lies in determining for the given fully controllable and fully observable plant (1) of the controller (2) satisfying the requirements on

- precision

$$|y_b(t)| \leqslant y^*, \tag{7}$$

- performance

$$t_{\text{reg}} \leqslant t^*_{\text{reg}}, \quad \sigma \leqslant \sigma^*, \tag{8}$$

- stability margins

$$r_a \geqslant r_a^*, \tag{9}$$

where y^*, t^*_{reg}, σ^*, and r^*_a are the given positive numbers.

Remark 2.1. As a matter of fact, the requirements on the system indices obey the inequalities

$$\begin{gathered} a_y y^* \leqslant |y(t)| \leqslant y^*, \quad a_t t^*_{\text{reg}} \leqslant t_{\text{reg}} \leqslant t^*_{\text{reg}}, \\ a_\sigma \sigma^* \leqslant \sigma \leqslant \sigma^*, \quad a_r r^*_a \leqslant r_a \leqslant r^*_a, \end{gathered} \tag{10}$$

where a_y, a_t, a_σ, and a_r are the given positive numbers ($a_y < 1$, $a_t < 1$, $a_\sigma < 1$ and $a_r < 1$) representing the tolerances on the deviations from the numbers y^*, t^*_{reg}, σ^*, and r^*_a. The point is that under conditions (7)–(9) it may happen that the calculations result in controller (2) providing the control error several times smaller than y^* for an exogenous disturbance close to the boundary f^*. Then, such precision of measuring the function $y(t)$ requires a hardware more precise then necessary.

A similar situation exists with the speed: if the control time is several times shorter than the required value t^*_{reg}, then the actuators are overloaded, system has noise-immunity, and so on. The further presentation is based on the aims (7)–(9), but it can be often developed for the case of (10).

2.2. Problem Reduction

By performing the Laplace transform of Eqs. (1), (2) under zero initial conditions, we represent them as

$$d(s)y = k(s)u + c(s)f, \tag{11}$$

$$g(s)u = r(s)y, \tag{12}$$

where

$$d(s) = \sum_{i=0}^{n} d_i s^i, \quad k(s) = \sum_{i=0}^{m} k_i s^i, \quad g(s) = \sum_{i=0}^{n_c} g_i s^i, \qquad r(s) = \sum_{i=0}^{m_c} r_i s^i, \quad c(s) = \sum_{i=0}^{p} c_i s^i.$$

We express requirements (7)–(9) using the system transfer function and its characteristic polynomial. The transfer function of system $t_{yf}(s)$ relating the output to the exogenous disturbance is given by

$$t_{yf}(s) = \frac{g(s)c(s)}{d(s)g(s) - k(s)r(s)}. \tag{13}$$

Requirement (7) on precision is satisfied if

$$\sup_{0 \leqslant \omega < \infty} |t_{yf}(j\omega)| \leqslant \frac{y^*}{f^*}. \tag{14}$$

Indeed, under disturbance (3) the output of system (1), (2) is given by (5).

Taking into account constraints (4), we put down

$$|y_b(t)| \leqslant \sum_{i=0}^{\infty} |a(\omega_i)| \leqslant \sum_{i=0}^{\infty} |t_{yf}(j\omega_i)||f_i| \leqslant \sup_{0 \leqslant \omega < \infty} |t_{yf}(j\omega)| \sum_{i=0}^{\infty} |f_i| = f^* \sup_{0 \leqslant \omega < \infty} |t_{yf}(j\omega)| \leqslant y^*,$$

and (14) follows from the last inequality.

The system characteristic polynomial is given by

$$d^s(s) = d(s)g(s) - k(s)r(s).$$

Assuming now that $n_c = n$, we order the absolute values of the real parts of the roots $s_{s,i}$, $i = \overline{1, 2n}$, of this polynomial:

$$|\mathrm{Re} s_{s,1}| \leqslant |\mathrm{Re} s_{s,2}| \leqslant \ldots \leqslant |\mathrm{Re} s_{s,2n}|.$$

The control time is characterized by the number

$$t_{\mathrm{reg}} = \beta |\mathrm{Re}\, s_{s,1}|^{-1},$$

where β is a positive number; $\beta = 3$ if $|\mathrm{Re}\, s_{s,2}|$ and the absolute values of the real parts of the roots of polynomials $g(s)$ and $c(s)$ in the transfer function (13) are sufficiently great as compared with the number $|\mathrm{Re}\, s_{s,1}|$.

We assume that requirement (8) on the time of control is satisfied if

$$|\mathrm{Re}\, s_{s,1}| \geqslant \beta t_{\mathrm{reg}}^{*-1}. \tag{15}$$

Expression (6) is expanded into

$$r_a^2 = \inf_{0 \leqslant \omega < \infty} |1 + w(j\omega)|^2 ,$$

where

$$w(s) = -\frac{k(s)r(s)}{d(s)g(s)}.$$

We assume that the system has stability margins if

$$r_a \geqslant r_a^*, \tag{16}$$

where, in particular, $r_a^* = 0.75$.

Let us consider the Bézout identity

$$d(s)g(s) - k(s)r(s) = \psi(s), \tag{17}$$

where $\psi(s)$ is the polynomial of degree $2n$ with roots having negative real parts.

For the controller polynomials $g(s)$ and $r(s)$, under $\deg d(s) = n$, $\deg r(s) = n - 1$ this identity has a unique solution which is established as that of the system of linear algebraic equations obtained by comparing the coefficients at the identical degrees s in the left and right sides of identity [13].

Problem 2.2 lies in determining the coefficients of the polynomial in the right side of identity (17) such that system (1), (2) satisfies conditions (14), (15), and (16).

Solution of Problem 2.2 is essentially dependent on the properties of the plant polynomial $k(s)$. We further assume that the roots of the polynomial $k(s)$ have negative real parts. In this case, (1) is called the minimum phase plant.

3. STRUCTURES OF THE POLYNOMIAL IN THE RIGHT SIDE OF THE BÉZOUT IDENTITY AND CONTROLLER POLYNOMIALS

We take the following structure of polynomial (17):

$$\psi(s) = \varepsilon(s)k(s)\delta(s), \tag{18}$$

where $\varepsilon(s)$ is the realizability polynomial of the degree $n-m-1$ required for controller realizability (the realizability condition $\deg g(s) \geqslant \deg r(s)$), $\delta(s)$ is the basic (main) polynomial of degree n with the real roots $(-s_{\delta,i})$, $i = \overline{1,n}$:

$$\delta(s) = \delta_n \prod_{i=1}^{n} (s + s_{\delta,i})$$

where we assume below that, if not otherwise stated,

$$\delta_n = d_n > 0. \tag{19}$$

If the degree of the polynomial $k(s)$

$$m = n - 1,$$

then the controller is realizable for $\varepsilon(s) = 1$, and the Bézout identity (17) has the evident solution

$$g(s) = k(s), \quad r(s) = d(s) - \delta(s).$$

If the degree of the polynomial $k(s)$ is less than $n-1$, then controller (2) is not realizable because the degree of the polynomial $r(s)$ is $n-1$. In this case, the realizability polynomial is given by

$$\varepsilon(s) = \varepsilon_\rho s^\rho + \ldots + \varepsilon_1 s + 1,$$

where $\rho = n - m - 1$ and the roots $\varepsilon(s)$ have negative real parts.

The structure of the polynomial $g(s)$ is given by

$$g(s) = g_\varepsilon(s) k(s) \tag{20}$$

where

$$g_\varepsilon(s) = g_{\varepsilon,\rho} s^\rho + \ldots + g_{\varepsilon,1} s + g_{\varepsilon,0}.$$

The coefficients of the realizability polynomial are selected so that

$$\varepsilon_i = \nu_i \varepsilon_{i-1}, \quad i = \overline{1,\rho}, \quad \varepsilon_0 = 1, \tag{21}$$

where ν_i, $i = \overline{1,\rho}$ are sufficiently small positive numbers such that the roots of the polynomial $\varepsilon(s)$ have negative real parts.

Condition (21) is satisfied if one assumes that

$$\varepsilon(s) = \prod_{i=1}^{\rho} \left(\frac{\mu_i}{s_\delta} s + 1 \right), \tag{22}$$

where $s_\delta = \max[s_{\delta,1}, \ldots, s_{\delta,n}]$, $i = \overline{1,\rho}$ and μ_i, $i = \overline{1,\rho}$ are sufficiently small different positive numbers.

Taking into consideration the structure of the modal polynomial and the polynomial $g(s)$, we represent the Bézout identity as

$$d(s) g_\varepsilon(s) - r(s) = \varepsilon(s) \delta(s).$$

Assertion 3.1. *For sufficiently small numbers ν_i, $i = \overline{1,\rho}$, the coefficients of the polynomials $g_\varepsilon(s)$ and $\tilde{r}(s)$ which are solution of the identity*

$$d(s) g_\varepsilon(s) - \tilde{r}(s) = \delta(s) \varepsilon(s),$$

are given by

$$g_{\varepsilon,i} = \varepsilon_i + 0_{1,\,i}(\nu), \quad \tilde{r}_j = r_j + 0_{2,\,j}(\nu), \quad i = \overline{1,\rho}, \quad j = \overline{0,n-1},$$

where $0_{1,\,i}(\nu)$ and $0_{2,\,j}(\nu)$ are the functions vanishing with the vector $\nu = [\nu_1, \ldots, \nu_\rho]$:

$$\lim_{\nu \to 0} 0_{1,\,i}(\nu) = 0, \quad \lim_{\nu \to 0} 0_{2,\,j}(\nu) = 0, \quad i = \overline{1,\rho}, \quad j = \overline{0,n-1}.$$

This and the subsequent assertions are proved in the Appendix.

4. DESIGN OF CONTROLLERS

4.1. Stability Margins

Using the recurrent difference

$$\mathrm{v}(s) = 1 + w(s),$$

we conclude that the system has stability margins if

$$|\mathrm{v}(j\omega)| \geqslant r_a^*, \quad 0 \leqslant \omega < \infty. \tag{23}$$

Divide the Bézout identity (17) by the polynomial $d(s)g(s)$ and find the following expression:

$$\mathrm{v}(s) = 1 + w(s) = \frac{\psi(s)}{d(s)g_\varepsilon(s)k(s)} = \frac{\varepsilon(s)\delta(s)}{g_\varepsilon(s)d(s)} = \frac{\varepsilon(s)\delta(s)}{[\varepsilon(s) + 0_1(s,\nu)]d(s)}.$$

Disregarding the polynomial $0_1(s,\nu)$ which vanishes together with the vector ν, we represent condition (23) as

$$\frac{|\delta(j\omega)|^2}{|d(j\omega)|^2} \geqslant r_a^*, \quad 0 \leqslant \omega < \infty, \tag{24}$$

which in the expanded form is given by

$$\frac{|\delta(j\omega)|^2}{|d(j\omega)|^2} = \frac{\prod\limits_{i=1}^{n_1}(\omega^2 + s_{\delta,i}^2)\prod\limits_{i=n_1+1}^{n_1+n_2}(\omega^2 + s_{\delta,i}^2)(\omega^2 + s_{\delta,i+1}^2)}{\prod\limits_{i=1}^{n_1}(\omega^2 + s_{d,i}^2)\prod\limits_{i=n_1+1}^{n_1+n_2}[\omega^4 + 2(2\xi_{d,i}^2 - 1)s_{d,i}^2\omega^2 + s_{d,i}^4]} \geqslant r_a^*, \tag{25}$$

where $(-\xi_{d,i} \pm j\sqrt{1-\xi_{d,i}^2})s_{d,i}$, $\xi_{d,i}^2 \leqslant 1$, $i = \overline{n_1+1, n_1+n_2}$, $(n_1 + 2n_2 = n)$ are complex roots, and $(-s_{d,i})$, $i = \overline{1, n_1}$, are real roots of the polynomial $d(s)$.

Assertion 4.1. *System* (11), (12) *with controller obtained from the Bézout identity* (17) *with polynomial* (18) *has the radius of stability margins $r_a = 1$ if the absolute value of the roots of the polynomial $\delta(s)$ satisfies the inequalities*

$$s_{\delta,i} \geqslant |s_{d,i}|, \quad i = \overline{1, n_1}, \tag{26}$$

$$s_{\delta,n_1+1} \geqslant |s_{d,n_1+1}|, \;\; s_{\delta,n_1+2} \geqslant |s_{d,n_1+1}|, \;\; \ldots, \;\; s_{\delta,n-1} \geqslant |s_{d,n}|, \;\; s_{\delta,n} \geqslant |s_{d,n}| \tag{27}$$

and the numbers μ_i, $i = \overline{1, n-m}$, in (22) *are sufficiently small.*

This assertion was obtained in [14] for the case where the polynomial $\delta(s)$ has as many complex roots as there are complex roots of the polynomial $d(s)$.

The sufficient condition for providing the stability margins

$$s_{\delta,i} = |s_{d,i}|(1 + \rho_i), \quad i = \overline{1, n}, \tag{28}$$

where ρ_i, $i = \overline{1, n}$ are arbitrary nonnegative numbers, follows from this assertion.

As compared with (28), less constructive yet necessary conditions for stability margins can be readily established from inequality (24).

Assertion 4.2. *To provide the system stability margins, it is necessary and sufficient the there be a polynomial $h(s)$ of degree n such that its roots have negative real parts such that*

$$|\delta(j\omega)|^2 - |d(j\omega)|^2 = |h(j\omega)|^2.$$

Indeed, as follows from condition (24) for $r_a^* = 1$,

$$a(j\omega) = |\delta(j\omega)|^2 - |d(j\omega)|^2 \geqslant 0, \quad 0 \leqslant \omega < \infty.$$

This inequality is satisfied if there exists the polynomial $h(s)$ such that it has the negative real parts and

$$a(j\omega) = h(-j\omega)h(j\omega) = |h(j\omega)|^2 \geqslant 0.$$

The inverse is true as well. If

$$|\delta(j\omega)|^2 - |d(j\omega)|^2 \geqslant |h(j\omega)|^2,$$

then $r_a = 1$.

Indeed, it follows from the last inequality that

$$\frac{|\delta(j\omega)|^2}{|d(j\omega)|^2} \geqslant 1 + \frac{|h(j\omega)|^2}{|d(j\omega)|^2}.$$

Since $\frac{|h(j\omega)|^2}{|d(j\omega)|^2} \geqslant 0$, condition (24) is satisfied.

4.2. Time of Control

Order the absolute values of the roots of the plant and basic polynomial as follows;

$$|s_{d,1}| \leqslant |s_{d,2}| \leqslant \cdots \leqslant |s_{d,n}|, \quad s_{\delta,1} \leqslant s_{\delta,2} \leqslant \cdots \leqslant s_{\delta,n}.$$

One can readily see that the requirement on the control time is satisfied if

$$s_{\delta,1} \geqslant \beta t_{\mathrm{reg}}^{*-1}.$$

To provide the control time, it suffices to take the numbers $\rho_i,\ i = \overline{1,n}$, in equalities (28) as

$$\rho_i = \frac{\beta t_{\mathrm{reg}}^{*-1}}{|s_{d,i}|} - 1, \quad i = \overline{1,n_1}, \quad \rho_i = 0, \quad i = \overline{n_1+1,n}, \tag{29}$$

where $n_1 \leqslant n$ is the number of the plant roots with absolute values of the real parts smaller than t_{reg}^{*-1}.

4.3. Control Precision and Overshoot for $c(s) = c_0$

It follows from (13), (17), (18), and (20) that for sufficiently small $\mu_i,\ i = \overline{1,\rho}$, in (22)

$$|t_{yf}(j\omega)|^2 = \frac{|c(j\omega)|^2}{|\delta(j\omega)|^2}.$$

In the case of $c(s) = c_0$,

$$|t_{yf}(j\omega)|^2 = \frac{c_0^2}{\delta_n^2 \prod\limits_{i=1}^{n} (\omega^2 + s_{\delta,i}^2)} \leqslant \frac{c_0^2}{\delta_n^2 \prod\limits_{i=1}^{n} s_{\delta,i}^2} = \frac{c_0^2}{\delta_0^2}.$$

We conclude from the last inequality and condition (14) that if

$$\delta_0 = \frac{|c_0| f^*}{y^*}, \tag{30}$$

then $\sup_{0 \leqslant \omega < \infty} |t_{yf}(j\omega)| \leqslant \frac{|c_0|}{\delta_0} = \frac{y^*}{f^*}$.

Thus, the following assertion is true.

Assertion 4.3. *The requirement on precision is satisfied if the free coefficient δ_0 of the polynomial $\delta(s)$ satisfies condition* (30) *and the numbers μ_i, $i = \overline{1, n-m}$, in* (22) *are small enough.*

The overshoot is given by

$$\sigma = 0.$$

Indeed, the system transfer function has the form $t_{yf}(s) = \frac{c_0}{\delta(s)}$ where the roots of the polynomial $\delta(s)$ are real which implies that there is no overshoot because according to [15] under the worst exogenous disturbance $\sup_{t_0 \leqslant t < t_1} |y(t)| = \frac{|c_0| f^*}{\delta_0}$.

4.4. Design Procedure

Procedure 4.1 of controller design consists of the following operations:

1. Determine the roots of the modal polynomial from (28), (29) which provides stability margins and the control time.

2. Compute the free coefficient of the polynomial $\delta(s)$:

$$\delta_0 = \delta_n \prod_{i=1}^{n} s_{\delta,i}$$

and compare it with the desired number in the right side of (30).

If this coefficient is less than this number, then

3. On the basis of (28), generate the values

$$s_{\delta,i} = |s_{d,i}|(1 + \rho_i q_m), \quad i = \overline{1, n},$$

where $q_m > 1$, and successively increase the number q_m until satisfying the condition of item 2.

4. Determine the polynomials of controller (12), by solving the Bézout identity $d(s)g(s) - k(s)r(s) = \varepsilon(s)k(s)\delta(s)$ where the polynomial $\varepsilon(s)$ like (22) has sufficiently small numbers μ_i, $i = \overline{1, n-m}$.

4.5. Control Precision in the General Case of the Disturbed Polynomial $c(s)$

In the general case, the polynomial $c(s)$ is given by

$$c(s) = c_p \prod_{i=1}^{p_1}(s + s_{c,i}) \prod_{i=p_1+1}^{p_1+p_2}(s^2 + 2\xi_{c,i} s_{c,i} + s_{c,i}^2),$$

where c_p is a number, $(-s_{c,i})$, $i = \overline{1, p_1}$, $p_1 + 2p_2 = n$ are real roots, and $(-\xi_{c,i} \pm j\sqrt{1 - \xi_{c,i}^2})s_{c,i}$, $|\xi_{c,i}| < 1$, $i = \overline{p_1 + 1, p + p_2}$, are complex roots.

In this case, for $d_n = \delta_n = 1$ the system frequency transfer function goes over to

$$|t_{yf}(j\omega)|^2 = \frac{c_p^2 \prod\limits_{i=1}^{p_1}(\omega^2 + s_{c,i}) \prod\limits_{i=p_1+1}^{p_1+p_2}[\omega^4 + 2(2\xi_{c,i}^2 - 1)s_{c,i}^2\omega^2 + s_{c,i}^4]}{\prod\limits_{i=1}^{p_1}(\omega^2 + s_{\delta,i}) \prod\limits_{i=p_1+1}^{p_1+p_2}(\omega^2 + s_{\delta,i}^2)(\omega^2 + s_{\delta,i+1}^2) \prod\limits_{i=p_1+p_2+1}^{n}(\omega^2 + s_{\delta,i}^2)}. \tag{31}$$

Let the inequalities

$$s_{c,i}^2 \leqslant s_{\delta,i}^2, \quad i = \overline{1, p_{11}}, \quad s_{c,i}^2 \geqslant s_{\delta,i}^2, \quad i = \overline{p_{11} + 1, p_1},$$
$$s_{c,i}^2 \leqslant s_{\delta,i}^2, \quad i = \overline{p_1 + 1, p_{22}}, \quad s_{c,i}^2 \geqslant s_{\delta,i}^2, \quad i = \overline{p_{22} + 1, p_1 + p_2}$$

be satisfied for the roots of the polynomials $c(s)$ and $\delta(s)$.

We represent the transfer function (31) as

$$|t_{yf}(j\omega)|^2 = c_p^2\beta_{11}(\omega)\beta_{12}(\omega)\beta_{21}(\omega)\beta_{22}(\omega)\beta_3(\omega),$$

where

$$\beta_{11}(\omega) = \frac{\prod\limits_{i=1}^{p_{11}}(\omega^2 + s_{c,i}^2)}{\prod\limits_{i=1}^{p_{11}}(\omega^2 + s_{\delta,i}^2)}, \quad \beta_{12}(\omega) = \frac{\prod\limits_{i=p_{11}+1}^{p_1}(\omega^2 + s_{c,i}^2)}{\prod\limits_{i=p_{11}+1}^{p_1}(\omega^2 + s_{\delta,i}^2)},$$

$$\beta_{21}(\omega) = \frac{\prod\limits_{i=p_1+1}^{p_{22}}[\omega^4 + 2(2\xi_{c,i}^2 - 1)s_{c,i}^2\omega^2 + s_{c,i}^4]}{\prod\limits_{i=p_1+1}^{p_{22}}[(\omega^2 + s_{\delta,i}^2)(\omega^2 + s_{\delta,i+1}^2)]},$$

$$\beta_{22}(\omega) = \frac{\prod\limits_{i=p_{22}+1}^{p_1+p_2}[\omega^4 + 2(2\xi_{c,i}^2 - 1)s_{c,i}^2\omega^2 + s_{c,i}^4]}{\prod\limits_{i=p_{22}+1}^{p_1+p_2}[(\omega^2 + s_{\delta,i}^2)(\omega^2 + s_{\delta,i+1}^2)]},$$

$$\beta_3(\omega) = \frac{c_p^2}{\prod\limits_{i=p_1+p_2+1}^{n}(\omega^2 + s_{\delta,i+1}^2)}.$$

Assertion 4.4. *If the condition*

$$\prod_{i=p_1+p_2+1}^{n} s_{\delta,i}^2 \geqslant \beta_{12}(0)\beta_{22}(0)\frac{f^{*2}}{y^{*2}}c_p^2 \tag{32}$$

is satisfied for $n - p$ roots of the basic polynomial, then the requirement on precision is satisfied.

5. CONCLUSIONS

For the minimum phase plants, the relation between the structure and coefficients of the right side of the Bézout identity with the indices (precision, speed, and stability margins) of the system whose controller is the solution of this identity was established. This relation underlies the method to design the controller providing the given values of the system indices in the absence of measurement errors.

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APPENDIX

Proof of Assertion 3.1. We represent the identity under consideration in an expanded form omitting the subscript ε in the polynomial $g_\varepsilon(s)$:

$$\left(d_n s^n + \sum_{i=0}^{n-1} d_i s^i\right)\left(\sum_{i=0}^{\rho} g_i s^i\right) - \sum_{i=0}^{n-1} \tilde{r}_i s^i = \left(\delta_n s^n + \sum_{i=0}^{n-1} \delta_i s^i\right)\left(\sum_{i=1}^{\rho} \varepsilon_i s^i + 1\right). \tag{A.1}$$

By comparing the coefficients at the degrees s from $n+\rho$ to n, we establish the following equation system for determination of the coefficients g_i, $i=\overline{0,\rho}$:

$$d_n g_\beta = \delta_n \varepsilon_\beta - \sum_{i=1}^{\rho-\beta} d_{n-i}\, g_{\beta+i} + \sum_{i=1}^{\rho-\beta} \delta_{n-i}\, \varepsilon_{\beta+i}, \quad \beta=\overline{0,\rho}. \tag{A.2}$$

Expression (21) for the coefficients of the realizability polynomial means that

$$\varepsilon_\rho \ll \varepsilon_{\rho-1} \ll \ldots \ll \varepsilon_2 \ll \varepsilon_1 \ll 1 \tag{A.3}$$

for sufficiently small ν_i, $i=\overline{1,\rho}$.

The right sides of Eqs. (A.2) contain the desired coefficients at the senior degrees of s, and the left sides, the coefficients at the degrees s smaller by one. Therefore, we conclude with regard for inequalities (A.3) and condition (19) that

$$g_\beta = \varepsilon_\beta + 0_{1,\beta}(\nu), \quad \beta=\overline{0,\rho}. \tag{A.4}$$

Now we compare the coefficients in identity (A.1) at the degrees s from $n-1$ to 0 and find the following system of equations for determination of the coefficients r_i, $i=\overline{0,n-1}$:

$$\tilde r_{n-\alpha} = d_{n-\alpha}\, g_0 - \delta_{n-\alpha}\, \varepsilon_0 + \sum_{i=1}^{n-\alpha} d_{n-\alpha-i}\, g_i - \sum_{i=1}^{n-\alpha} \delta_{n-\alpha-i}\varepsilon_i, \quad \alpha=\overline{1,n}.$$

Since g_0 is close to 1 ($\varepsilon_0=1$) and the rest of the coefficients at d_i and δ_i, $i=\overline{1,n-\alpha}$, can be made arbitrarily small, we obtain $\tilde r_j = r_j + 0_{2,j}(\nu)$, $j=\overline{0,n-1}$, with regard for (A.3) and (A.4)

Proof of Assertion 4.1. We consider two kinds of functions making up the left side of inequality (25):

$$a_1(\omega)=\frac{\omega^2+a_1^2}{\omega^2+b_1^2}, \quad a_2(\omega)=\frac{\omega^4+2a_2^2c_1\omega^2+a_2^4}{\omega^4+2b_2^2c_2\omega^2+b_2^4}. \tag{A.5}$$

These functions have the following almost evident properties.

Property 1. If $a_1^2 \geqslant b_1^2$, then $a_1(\omega)\geqslant 1$, $0\leqslant\omega<\infty$.

Property 2. If $a_2^2\geqslant b_2^2$, $c_1\geqslant 1$, $-1\leqslant c_2\leqslant 1$, then $a_2(\omega)\geqslant 1$, $0\leqslant \omega<\infty$.

With their use properties we consider the transfer function (25). It follows from Property 1 and inequality (26) that

$$\frac{|\delta(j\omega)|^2}{|d(j\omega)|^2} \geqslant \frac{\prod\limits_{i=n_1+1}^{n_1+n_2}[\omega^4+(s_{\delta,i}^2+s_{\delta,i+1}^2)\omega^2+s_{\delta,i}^2 s_{\delta,i+1}^2]}{\prod\limits_{i=n_1+1}^{n_1+n_2}[\omega^4+2(2\xi_{d,i}^2-1)s_{d,i}^2\omega^2+s_{d,i}^4]} = \frac{\prod\limits_{i=n_1+1}^{n_1+n_2}[\omega^4+2c_{1,i}s_{\delta,i}^2\omega^2+s_{\delta,i}^2 s_{\delta,i+1}^2]}{\prod\limits_{i=n_1+1}^{n_1+n_2}[\omega^4+2c_{2,i}s_{d,i}^2\omega^2+s_{d,i}^4]},$$

where

$$c_{1,i}=\frac{s_{\delta,i}^2+s_{\delta,i+1}^2}{2s_{\delta,i}^2}, \quad c_{2,i}=2\xi_{d,i}^2-1, \quad i=\overline{n_1+1,n_1+n_2}. \tag{A.6}$$

One can easily see that $c_{1,i}\geqslant 1$ because $s_{\delta,i+1}\geqslant s_{\delta,i}$. Since $|\xi_{d,i}^2|\leqslant 1$, we have $-1\leqslant c_{2,i}\leqslant 1$, and so under conditions (26) and (27) Property 2 is valid. Therefore,

$$\frac{|\delta(j\omega)|^2}{|d(j\omega)|^2}\geqslant 1.$$

Proof of Assertion 4.4. We continue to study functions (A.5).

Property 3. If $a_1^2 \leqslant b_1^2$, then $a_1(\omega) \leqslant 1,\ 0 \leqslant \omega < \infty$.

Proof. Its proof follows the lines of Property 1.

Property 4. If $a_2^2 \leqslant b_2^2$, $-1 \leqslant c_1 \leqslant 1$, $c_2 > 1$, then $a_2(\omega) \leqslant 1,\ 0 \leqslant \omega < \infty$.

Proof. The expression

$$b_2^4 - a_2^4 + 2(b_2^2 c_2 - a_2^2 c_1)\omega^2 \geqslant 0,$$

follows from this inequality. It is satisfied if

$$b_2^2 \geqslant a_2^2, \quad b_2^2 c_2 \geqslant a_2^2 c_1.$$

If notation similar to (A.6) is introduced, then it follows from these properties

$$\beta_{11}(\omega) \leqslant 1, \quad \beta_{21}(\omega) \leqslant 1, \quad 0 \leqslant \omega < \infty.$$

Properties 1 and 2 define the lower bounds of the functions (A.5). Now we determine their upper bounds.

Property 5. If $a_1^2 \geqslant b_1^2$, then

$$a_1(\omega) \leqslant \frac{a_1^2}{b_1^2}, \quad 0 \leqslant \omega < \infty. \tag{A.7}$$

Proof. With regard for definition (A.5), we present inequality (A.7) as

$$b_1^2\omega^2 + b_1^2 a_1^2 \leqslant a_1^2\omega^2 + a_1^2 b_1^2$$

which provides $a_1^2 \geqslant b_1^2$. Inversely, inequality (A.7) follows from the latter one.

Property 6. If $a_2^2 \geqslant b_2^2$, $-1 \leqslant c_1 \leqslant 1, c_2 > 1$, then $a_2(\omega) \leqslant \frac{a_2^4}{b_2^4}$.

Proof. Represent the inequality as

$$b_2^4\omega^4 + 2a_2^2 b_2^4 c_1\omega^2 + a_2^4 b_2^4 \leqslant a_2^4 b_2^4 + 2a_2^4 b_2^2 c_2\omega^2 + a_2^4\omega^4,$$

or

$$(a_2^4 - b_2^4)\omega^4 + 2a_2^2 b_2^2(a_2^2 c_2 - b_2^2 c_1)\omega^2 \geqslant 0.$$

The latter inequality is satisfied because $a_2^2 \geqslant b_2^2$ and $a_2^2 c_2 > b_2^2 c_1$.

It follows from Properties 5 and 6 that

$$\beta_{12}(\omega) \leqslant \frac{\prod\limits_{i=p_{11}+1}^{p_1} s_{c,i}^2}{\prod\limits_{i=p_{11}+1}^{p_1} s_{\delta,i}^2} = \beta_{12}(0), \quad \beta_{22}(\omega) \leqslant \frac{\prod\limits_{i=p_{22}+1}^{p_1+p_2} s_{c,i}^4}{\prod\limits_{i=p_{22}+1}^{p_1+p_2} s_{\delta,i}^2 s_{\delta,i+1}^2} = \beta_{22}(0).$$

Therefore,

$$|t_{yf}(j\omega)|^2 \leqslant \beta_{12}(0)\beta_{22}(0)\beta_3(\omega) \leqslant \beta_{12}(0)\beta_{22}(0)\beta_3(0)$$

and then inequality (32) of the statement follows from the inequality

$$\beta_{12}(0)\beta_{22}(0)\beta_3(0) \leqslant \frac{y^{*2}}{f^{*2}}.$$

REFERENCES

1. *Osnovy avtomaticheskogo regulirovaniya* (Fundamentals of Automatic Control), Solodovnikov, V.V., Ed., Moscow: Mashgiz, 1954.

2. Voronov, A.A., *Osnovy teorii avtomaticheskogo upravleniya. Ch. I* (Fundamentals of the Automatic Control Theory. Part I), Moscow: Energiya, 1965.

3. Letov, A.M., Analytical Controller Design. I–IV, *Autom. Remote Control*, 1960, vol. 21, no. 4, pp. 303–306; no. 5, pp. 389–393; no. 6, pp. 458–461; 1961, vol. 22, no. 4, pp. 363–372.

4. Kalman, R.E., Contribution to the Theory of Optimal Control, *Bullet. Soc. Mat. Mech.*, 1960, vol. 5, no. 1, pp. 102–119.

5. Zames, G., Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses, *IEEE Trans. Automat. Control*, 1981, vol. 26, no. 2, pp. 301–320.

6. Doyle, J.C., Glover, K., Khargonekar, P.P., and Francis, B.A., State-Space Solution to Standard H_2 and H_∞ Control Problem, *IEEE Trans. Automat. Control*, 1989, vol. 34, no. 8, pp. 831–846.

7. Aleksandrov, A.G. and Nebaluev, N.A., Analytical Synthesis of Controller Transfer Matrices on the Basis of Frequency Domain Performance Criteria. I, *Autom. Remote Control*, 1971, vol. 32, no. 12, part 1, pp. 1871–1878.

8. Safonov, M.G. and Athans, M., Gain and Phase Margin for Multi-loop LQG Regulators, *IEEE Trans. Automat. Control*, 1977, vol. AC-22, no. 2, pp. 173–179.

9. Aleksandrov, A.G., *Metody postroeniya sistem avtomaticheskogo regulirovaniya* (Methods of Designing the Automatic Control Systems), Moscow: Fizmatlit, 2008.

10. Aleksandrov, A.G., On Analytical Design of Controllers, *Autom. Remote Control*, 2010, vol. 71, no. 6, pp. 977–992.

11. Chestnov, V.N. and Zatsepilova, Zh.V., Design of Multidimensional Systems by the Engineering Indices of Precision, Control Time, and Stability Margins, in *Collected Papers Int. Conf. "Problems of Information Control, Transmission, and Processing" (ATM-TKI-50)*, Saratov: SGTU, 2009, pp. 41–45.

12. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh* (Control of Linear Systems under Exogenous Disturbances), Moscow: URSS, 2014.

13. *Adaptivnoe upravlenie dinamicheskimi ob"ektami* (Adaptive Control of Dynamic Plants), Fomin, V.N., Fradkov, A.L., and Yakubovich, V.A., Eds., Moscow: Nauka, 1981.

14. Chestnov, V.N., Modal Control of One-dimensional Plants with Allowance for the Given Radius of the Stability Margins, in *Proc. Int. Conf. "Analytical Theory of Automatic Control and its Applications,"* Saratov: SGTU, 2000, pp. 159–164.

15. Chebotarev, N.G., On a Mathematical Problem Arising in Connection with Estimation of Deviations of the Controlled Variable if the Disturbing Force is bounded in the Absolute Value, *Avtom. Telemekh.*, 1948, vol. 9, no 4, pp. 331–334.

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