TOWARD FINITE-FREQUENCY THEORY OF CONTROL

A.G. ALEXANDROV

Depart. of Automation of Manufacturing, Moscow Institute of Steel and Alloys, Pervomayskaya 7, Elektrostal, Moskowskaya Oblast, 144000, Russia.

Abstract. Taking to account application matter it is suggested the finite-frequency conception of the linear plant control. The concept bases self upon a notion of the frequency domain parameters as values measuring in experiment and completely describing the plant.

It is shown that these parameters are sufficiently for a analysis of controllability and stability, the controller and adaptive control design. The methods and algorithms are given.

Keywords. Frequency response, controllability, stability, controller design, adaptive control.

1. INTRODUCTION

The last years the design time of a plant and control device is strongly decreased. This is the result of using of computer-aided-design (CAD) and unification of the control device elements. Shortening of a design duration creates often a situation when the plant and control system are ready to putting into operation but there is not the mathematical model of a plant and therefore the control algorithm cannot be found.

In these conditions it is necessary a new approach proceeding from the experimental investigations results rather then the mathematical model of plant.

The purpose of this paper is the finite-frequency conception in which the plant is described by the frequency domain parameters (FDP). The FDP are derived experimentally under the plan testing by the harmonic signals.

The conception name reflects its connection with a classical frequency control theory (MacFarlane, 1979) and its distinction that consists in using of finite number of the test frequencies.

2. FREQUENCY DOMAIN PARAMETERS (FDP)

Consider the plant described by the equation

$$y^{(n)}+d_{n-1}y^{(n-1)}+\ldots+d_1\dot{y}+d_0y=k_{n_1}u^{(n_1)}+\ldots+$$

$$+k_0u+m_{n_2}f^{(n_2)}+\ldots+m_0f \quad (n_1<n,\ n_2<n), \qquad (2.1)$$

in which $y(t)$ is the measured output, $u(t)$ is the controlled input, $f(t)$ is the external disturbance.

The coefficients d_i, k_j, m_k $(i=\overline{0,n-1}, j=\overline{0,n_1}, k=\overline{0,n_2})$ are unknown. The external disturbance is a bounded function

$$|f(t)|\le f^* , \qquad (2.2)$$

where f^* is the specified number.

It is assumed that the plant (2.1) is completely controllable. If the plant is unstable the estimation C_0 of the unstability degree

$$C_0>\max\{\mathrm{Re}\ s_1,\ldots,\mathrm{Re}\ s_n\}, \qquad (2.3)$$

where s_i $(i=1,n)$ are the roots of polynomial $d(s)=s^n+d_{n-1}s^{n-1}+\ldots+ +d_0$ is known.

Definition 2.1 (Alexandrov, 1989). A set of the 2n numbers

$$\alpha_k=\mathrm{Re}\ W(\lambda+j\omega_k),\beta_k=\mathrm{Im}\ W(\lambda+j\omega_k)(k=\overline{1,n}),\ \lambda>C_0 \qquad (2.4)$$

is called the frequency domain parameters (FDP).

Here $W(s)=k(s)/d(s)$ is the transfer function of the plant (2.1), ω_k $(k=\overline{1,n})$ are the positive numbers that are below called the test frequencies, $\omega_i\neq0$, $\omega_i\neq\omega_j$, $i\neq j$, $(i, j=\overline{1,n})$. ■

The method of the experimental determination of the FDP consists in the following.

Apply to the input of the plant (2.1) the test signal

$$v(t)=e^{\lambda t}\sum_{i=1}^{n}\rho_k\sin\omega_k t\ ,\quad \lambda>C_0, \qquad (2.5)$$

where λ, ω_k, ρ_k $(k=\overline{1,n})$ are given numbers.

The signal $y(t)$ after multiplication by $e^{-\lambda t}$ is applied to the input of the Fourier filter(Eykhoff, 1974). The outputs of the filter give the FDP estimations

$$\alpha_k(\delta)=\frac{2}{\rho_k\delta}\int_{t_0}^{t_0+\delta}y(t)e^{-\lambda t}\sin\omega_k t\ dt,\ (k=\overline{1,n}) \qquad (2.6)$$

$$\beta_k(\delta)=\frac{2}{\rho_k\delta}\int_{t_0}^{t_0+\delta}y(t)e^{-\lambda t}\cos\omega_k t\ dt,\ (k=\overline{1,n}) \qquad (2.7)$$

Theorem 2.1. The estimations of the frequency domain parameters of the plant (2.1) have the next property

$$\lim_{\delta\to\infty}\alpha_k(\delta)=\alpha_k,\quad \lim_{\delta\to\infty}\beta_k(\delta)=\beta_k,\quad (k=\overline{1,n}) \qquad (2.8)$$ ■

The theorem proof is given in Appendix.

Definition 2.2. A set of the $2n_3$ numbers (2.4) where $k=\overline{1,n_3}$, and $n\le n_3\le 2n$ is called the widened frequency domain parameters of the plant (2.1). ■

3. PROBLEMS STATEMENT

Consider the system (2.1) with a controller

$$g_{n-1}u^{(n-1)}+\ldots+g_0u=r_{n-1}y^{(n-1)}+\ldots+r_0y \qquad (3.1)$$

Problem A. Find the conditions of the output boundness of the system (2.1), (3.1) if the plant is described by widened FDP and the controller coefficients are known. ∎

Let some complementary apriory information about the polynomials k(s) and m(s) of the plant (2.1) and the external disturbance be:

(a) k(s) is the Hurwitz polynomial (its roots have the negative real parts);

(b) coefficient m_0 and the least root $\bar{s}$ of the polynomial m(s) satisfy the next conditions $|m_0|\le m_0^*$, $|\bar{s}|\ge \bar{s}^*$ where m_0^* and $\bar{s}^*$ are the specified numbers;

(c) the external disturbance may be represented as

$$f(t)=\sum_{i=1}^{\gamma_1}\delta_i^s\sin\omega_i^f t+\sum_{j=1}^{\gamma_2}\delta_j^c\cos\omega_j^f t\ , \qquad (3.2)$$

where δ_i^s $(i=\overline{1,\gamma_1})$, δ_i^c $(i=\overline{1,\gamma_2})$ and ω_i^f $(i=\overline{1,\max(\gamma_1,\gamma_2)})$ are the unknown numbers.

Problem B (accurate control). Find the coefficients of the controller (3.1) for the plant (2.1) described by the FDP α_k and β_k $(k=\overline{1,n})$ such that the system (2.1), (3.1) satisfies the requirements on accuracy

$$|y(t)|\le y^*\ ,\quad t\ge\bar{t}\ , \qquad (3.3)$$

where y^* is given number, $\bar{t}$ is a time moment in which transients damp out. ∎

If the coefficients of the plant (2.1) are known the coefficients of the controller (3.1) may be found using the Bezout-Identity

$$d(s)g(s)-k(s)r(s)=x(s)k(s)\tilde{\psi}(s) \qquad (3.4)$$

where Hurvitz polynomials $x(s)$ and $\tilde{\psi}(s)$ are built up (Alexandrov, 1992) so that the controller to be found from the identity (3.4) complies with the requirements on accuracy (3.3). ∎

If there is not the information about the polynomial k(s) and m(s), then the following more weak task arises.

Problem C (modal control). Find the coefficients of the controller (3.1) for plant (2.1) described by the FDP α_k and β_k $(k=\overline{1,n})$ such that the characteristic polynomial of system (2.1), (3.1) coincides with the given Hurwitz polynomial $\delta(s)$. ∎

If the coefficients of the plant are known the solution of problem C consists in the solution of the next Bezout-Identity

$$d(s)g(s)-k(s)r(s)=\delta(s) \qquad (3.5)$$

Solution of the problem C for the unknown coefficients is built up from two steps: the first step is the identification of these coefficients using the given FDP of plant, the second step is the solution of identity (3.5).

The next task came into existence in connection with the first step.

Problem D (identification). Find the coefficients of the plant (2.1) using its FDP. ∎

Let us consider the assumptions of the section 2 about the plant properties.

Problem E (controllability analysis). Find the conditions of the controllability of the plant described by FDP α_k and β_k $(k=\overline{1,n})$. ∎

In the problems B and C it is implied that the FDP determination and control design are separated in the time. Now we'll assume that the FDP are unknown and found during the process of the controller tuning.

Problem F. (adaptive accuracy control). Find the coefficients tuning algorithm of the controller (3.1) such that in the presence of external disturbance (3.2) the requirement on accuracy is fulfilled

$$|y(t)|\le y^*\ ,\quad t\ge\tilde{\bar{t}} \qquad (3.6)$$

where $\tilde{\bar{t}}$ is a time moment. ∎

Time moment $\tilde{\bar{t}}$ includes both the damped time and the controller tuning time.

4. FREQUENCY EQUATIONS

Consider Bezout-Identity

$$a(s)c(s)-b(s)e(s)=q(s) \qquad (4.1)$$

where $a(s)=s^p+\sum_{i=0}^{p-1}a_is^i$, $b(s)=\sum_{i=0}^{p_1}b_is^i$, $(p>p_1)$ and $q(s)=\sum_{i=0}^{p_2}q_is^i$ are the given polynomials with the real coefficients, $c(s)=\sum_{i=0}^{p_3}c_is^i$, $e(s)=\sum_{i=0}^{p-1}e_is^i$ are the sought polynomials with the real coefficients.

Let us divide the identity (4.1) by some polynomial v(s) of the degree p and then we obtain

$$W_1(s)c(s)-W_2(s)e(s)=W_3(s) \qquad (4.2)$$

where

$$W_1(s)=a(s)/v(s),\ W_2(s)=b(s)/v(s),\ W_3(s)=q(s)/v(s).$$

Assuming that

$$v(s_k)\ne0\quad (k=\overline{1,p}) \qquad (4.3)$$

we place in (4.2) $s=s_k=\lambda+j\omega_k$ $(k=\overline{1,p})$, and derive the *frequency equations*

$$(\alpha_{1k}+j\beta_{1k})c(s_k)-(\alpha_{2k}+j\beta_{2k})e(s_k)=(\alpha_{3k}+j\beta_{3k})$$
$$(k=\overline{1,p}) \qquad (4.4)$$

in which $\alpha_{ik}=\mathrm{Re}\ W_i(s_k)$, $\beta_{ik}=\mathrm{Im}\ W_i(s_k)$ $(i=1,2,3,$ $k=\overline{1,p})$ are the frequency domain parameters of the transfer functions $W_i(s)$ $(i=1,2,3)$.

Lemma 4.1. (Alexandrov, 1989) The frequency equations (4.4) have the unique solution which coincides with the solution of (4.1) if the polynomials a(s) and b(s) are relatively prime and the frequencies $\omega_k\ne0$, $\omega_k\ne\omega_i$ $(k\ne i)$ $(i,k=\overline{1,p})$. ∎

5. STABILITY CRITERION

The characteristic polynomial of system (2.1),(3.1) has a view

$$n(s)=d(s)g(s)-k(s)r(s) \qquad (5.1)$$

Consider the identity of a view (4.1)

$$n(s)c(s)-k(s)e(s)=s^{2n-1}k(s) \qquad (5.2)$$

under p=2n-1.

It is easily shown that the polynomials n(s) and k(s) are relatively prime and then identity (5.2) has the unique solution $e(s)=n(s)-s^{2n-1}$, $c(s)=k(s)$. If the identity (5.1) divides by the polynomial d(s) and places $s=s_k=\lambda+j\omega_k$ $(k=\overline{1,2n-1})$ we derive the expression (4.2) in which

$W_1(s_k)=g(s_k)-W(s_k)r(s_k)$, $W_2(s_k)=W(s_k)$,

$$W_3(s_k)=s_k^{2n-1}W(s_k). \quad (k=\overline{1,2n-1}) \qquad (5.3)$$

Criterion 5.1.(Alexandrov, 1991a) The system (2.1), (3.1) with a plant described by the widened FDP is asymptotically stable under f(t)=0 (and has the bounded solutions if f(t) is a bounded function) if and only if the solution of the frequency equations (4.4) (5.3) satisfies for e_i $(i=\overline{0,2n-1})$ the inequalities of Hurwitz's criterion of stability.

6. CONTROLLER DESIGN

Accurate Control of a Minimum-Phased Plant (Solu tion of the Problem B)

Dividing the identity (3.4) by polynomial d(s) and placing $s=s_k=\lambda+j\omega_k$ one can obtain the expression (4.4) in which

$$W_1(s_k)=1,\ W_2(s_k)=W(s_k),\ W_3(s_k)=W(s_k)\times(s_k)\tilde{\Psi}(s_k). \quad (k=\overline{1,n}) \qquad (6.1)$$

So, if the coefficients of the controller (3.1)

$$g_i=c_i,\ r_i=e_i \quad (i=\overline{0,n-1}), \qquad (6.2)$$

where c_i and e_i $(i=\overline{0,n})$ are the solution of the frequency equations (4.4), (6.1) then the system (2.1), (3.1) satisfies the requirement to accuracy (3.3).

Identification (Solution of the Problem D)

Form the identity

$$d(s)c(s)-k(s)e(s)=s^n k(s)\ , \qquad (6.3)$$

which has the unique solution $s^n+e(s)=d(s)$, $c(s)=k(s)$.

Dividing the identity (6.3) by d(s) and placing $s=s_k=\lambda+j\omega_k$ one can obtain the expression (4.4) in which

$$W_1(s_k)=1,\ W_2(s_k)=W(s_k),\ W_3(s_k)=s_k^n W(s_k),\ (k=\overline{1,n}) \qquad (6.4)$$

So, the coefficients of the plant (2.1)

$$k_i=c_i\ (i=\overline{0,n_1}),\ d_j=e_j\ (j=\overline{0,n-1}). \qquad (6.5)$$

where c_i and e_i $(i=\overline{1,n})$ are the solution of the frequency equations (4.4), (6.4).

Modal Control of Nonminimum-Phased Plant (Solution of the Problem C)

Using the values (6.5) we find the coefficients of the controller (3.1) from (3.5).

Completely Controllability Condition (Solution of the Problem E)

Criterion 6.1. The plant (2.1) described the FDP is completely controllability if and only if the determinant of the matrix of the frequency equation (4.4), (6.1) is not zero.

7. ADAPTIVE CONTROL (SOLUTION OF THE PROBLEM F)

Process of the controller design in the accurate control problem (problem B) may be separated in a time from a process finding of FDP. These processes must be continuous or simultaneous in the adaptive systems.

Algorithm 7.1. It contains the following steps:

Step 1. Apply the test signal (2.5) to the plant (2.1) and obtain the FDP estimations $\alpha_k(\delta_1)$ and $\beta_k(\delta_1)$ $(k=\overline{1,n})$ with Fourier filter output. A filtring interval δ_1 is determined from the necessary conditions of convergence

$$|\alpha_k(\delta_1)-\alpha_k(\delta_1-\Delta t)|\le\epsilon_p\ ,\ |\beta_k(\delta_1)-\beta_k(\delta_1-\Delta t)|\le\epsilon_p \quad (k=\overline{1,n}), \qquad (7.1)$$

where ϵ_p is a sufficiently small number, Δt is a number.

Step 2. Solve the system (4.4), (6.1) replacing α_k and β_k $(k=\overline{1,n})$ by their estimations and find the coefficients of the controller $g_i(\delta_1)$ and $r_i(\delta_1)$ $(i=\overline{1,n-1})$.

Step 3. Close the plant (2.1) by the controller

$$g(\delta_1,s)u=r(\delta_1,s)y+l(s)v(t), \qquad (7.2)$$

where $l(s)=l_q s^q+\ldots+l_0$, $q<n$, l_i $(i=\overline{0,q})$ are a given numbers and using Fourier filter obtain the FDP of the system (2.1), (7.2) $\nu_k(\delta_2)$ and $\mu_k(\delta_2)$ $(k=\overline{1,n})$ where a filtring interval δ_2 is determined from the necessary conditions of convergence

$$|\nu_k(\delta_2)-\nu_k(\delta_2-\Delta t)|\le\epsilon_c\ ,\ |\mu_k(\delta_2)-\mu_k(\delta_2-\Delta t)|\le\epsilon_c, \quad (k=\overline{1,n}) \qquad (7.3)$$

where ϵ_c is a sufficiently small number.

Remark 7.1. Repeating of the proof of the theorem 2.1 one can obtain

$$\lim_{\delta_2\to\infty}\nu_k(\delta_2)=\nu_k=\mathrm{Re}\ W_{c1}(s_k),$$
$$\lim_{\delta_2\to\infty}\mu_k(\delta_2)=\mu_k=\mathrm{Im}\ W_{c1}(s_k), \quad (k=\overline{1,n}) \qquad (7.4)$$

where

$$W_{c1}(s)=\frac{W_1(s)W(s)}{1-W(s)W_c(s)}\ , \qquad (7.5)$$

$W_1(s)=l(s)/g(s)$, $W_c(s)=r(s)/g(s)$.

Remark 7.2. The following notion is used below to determine a moment of adaptation finishing. Let us introduce the function

$$\eta(\delta_2)=\sqrt{\sum_{k=1}^{n}\gamma_k^{(1)}(\bar{\nu}_k^*-\bar{\nu}_k(\delta_2))^2+\sum_{k=1}^{n}\gamma_k^{(2)}(\bar{\mu}_k^*-\bar{\mu}_k(\delta_2))^2} \qquad (7.6)$$

where $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ $(k=\overline{1,n})$ are the given positive numbers,

$$\bar{\nu}_k(\delta_2)=\nu_k(\delta_2)q_k^{-1}(\delta_2)\ ,\ \bar{\mu}_k(\delta_2)=-\mu_k(\delta_2)q_k^{-1}(\delta_2),$$
$$q_k(\delta_2)=\nu_k^2(\delta_2)+\mu_k^2(\delta_2)\ ,\ \bar{\nu}_k^*=\nu_k^* q_k^{*-1}\ ,\ \bar{\mu}^*=-\mu_k^* q_k^{*-1},$$
$$q_k^*=\nu_k^{*2}+\mu_k^{*2}\ , \quad (k=\overline{1,n}),$$

ν_k^* and μ_k^* $(k=\overline{1,n})$ *are the FDP of a closed-loop* system with "ideal" controller (6.2).

A number $\eta(\delta_i)$ is the index of aproximity of the controllers (7.2) to "ideal" controller. The FDP ν_k^* and μ_k^* $(k=\overline{1,n})$ may be calculated apriori. In fact, taking account the identity (3.4) we obtain from (7.5) the transfer function $W_{c1}^*(s)=l(s)/\times(s)\tilde{\psi}(s)$ with the known coefficients. ■

Step 4. Check the condition of proximity

$$\eta(\delta_2)\le\epsilon, \qquad (7.7)$$

(ϵ is the given number). It may be shown that (7.7) is a sufficiently condition of adaptation convergence.

If (7.7) is fulfilled, place v(t)=0 and the end.

Step 5. Calculate the FDP estimations of the plant

$\alpha(\delta_2)$ and $\beta(\delta_2)$ from the formulae (7.5) under $s=s_k$ $(k=\overline{1,n})$ placing $\nu_k=\nu_k(\delta_2)$, $\mu_k=\mu_k(\delta_2)$ $(k=\overline{1,n})$, $g(s)=g(\delta_1,s)$, $r(s)=r(\delta_1,s)$ and solve the system (4.4), (6.1) substituting $\alpha_k=\alpha(\delta_2)$ and $\beta_k=\beta(\delta_2)$ $(k=\overline{1,n})$ and go to step 3 (substituting the coefficients of the controller (7.2) by one to be obtained) and so on.

Remark 7.3. The filtering interval is now determined as

$$\delta_3=\max(\tilde{\delta}_3,\delta_2)+K \tag{7.8}$$

where $\tilde{\delta}_3$ is found from (7.3), K is a positive number. ■

The adaptation process converges if (a) the filtring intervals durations satisfy the conditions of view (7.8) and (b) the unstability degrees of the system for each filtring interval are least or equal than C_0.

If the condition (b) is not fulfilled then it is necessary to increase a number λ in the test signal and the Fourier filter.

8. CONCLUSION

The finite-frequency theory of a control of the single-input-single-output (SISO) plants is constructed.

The classical frequency control theory originated from Nyquist's criterion and Bode diagram is the strongly instrument of the control system design more fifty years already. In some sense the finite-frequency theory is the modification of the classical frequency theory. Such modification is challenged by a change of the control system design conditions.

9. REFERENCE

Alexandrov A.G.(1989) Method of the frequency domain parameters. *Automation and Remote Control*, vol. 50, N°12.

Alexandrov A.G. (1991a) Finite-frequency criterions of stability of automatic control system with uncertain parameters. *Proceeding of the first European Control Conference*, Grenoble, France, pp. 2591-2596.

Alexandrov A.G. (1991b) Frequency controllers. *Automation and Remote Control*. vol.52, N°1.

Alexandrov A.G. (1992) Frequency adaptive control. *Preprints of the fourth IFAC international symposium on adaptive systems in control and signal processing*. France, pp. 47-52.

Demidovich. B.P. (1967) *Lectures on mathematical theory of stability*. M., Nauka. (in Russian).

Eykhoff P. (1974) *System identification parameter and state estimations*, Y. Wiley and Sons. Ltd.

MacFarlane A.G.J. (1979) The development of frequency response method in automatic control. *IEEE Trans. Automat. Control.*, vol. AC-24 N°1, pp. 250-265.

Volovitch W.A. (1974) *Linear multivariable systems*. Springer-Verlag.

APPENDIX

The proof of the theorem 2.1.

Transform the equation (2.1) to the Cauchy form

$$\dot{x}=Ax+bu+cf,\quad y=dx\ , \tag{A.1}$$

where $x(t)$ is the n dimensional vector of the state variable of the plant, A, b, c and d are a matrix and vectors of the numbers respectively.

Solution of the equation (A.1) has a view

$$y(t)=x_1(t)+x_2(t)+x_3(t) \tag{A.2}$$

where

$$x_1(t)=de^{A(t-t_0)}x(t_0),\quad x_2(t)=d\int_{t_0}^{t} e^{A(t-\tau)}bu(\tau)\,d\tau,$$

$$x_3(t)=d\int_{t_0}^{t} e^{A(t-\tau)}cf(\tau)\,d\tau. \tag{A.3}$$

If $u(t)=v(t)$, and $f(t)=0$, then (Alexandrov, 1989, Alexandrov, 1991b) the relations (2.6), (2.7) are fulfilled.

Therefore, to prove the theorem we must show that the expressions

$$e_{\alpha k}(\delta)=\frac{2}{\rho\delta}\int_{t_0}^{t_0+\delta} x_3(t)e^{-\lambda t}\sin\omega_k t\,dt,$$

$$e_{\beta k}(\delta)=\frac{2}{\rho\delta}\int_{t_0}^{t_0+\delta} x_3(t)e^{-\lambda t}\cos\omega_k t\,dt,\ (k=\overline{1,n}) \tag{A.4}$$

are the vanishing functions:

$$\lim_{\delta\to\infty} e_{\alpha k}(\delta)=0,\quad \lim_{\delta\to\infty} e_{\beta k}(\delta)=0,\quad (k=\overline{1,n}). \tag{A.5}$$

It is obvious, that

$$|x_3(t)|=\left|\int_{t_0}^{t} h(t-\tau)f(\tau)\,d\tau\right|\le f^*\int_{t_0}^{t}|h(t-\tau)|\,d\tau, \tag{A.6}$$

where $h(t-\tau)=de^{A(t-\tau)}c$.

It is known (Demidovich, 1967), that

$$|h(t-\tau)|\le k_1e^{(s^*+\varepsilon)(t-\tau)},\ t>\tau \tag{A.7}$$

where $s^*=\max\limits_{1<i<n}\operatorname{Re}\lambda_i(A)$, and $\lambda_i(A)$ $(i=\overline{1,n})$ are the eigenvalues of the matrix A, $\varepsilon>0$ is any sufficiently small number, k_1 is a number.

So,

$$|x_3(t)|\le\frac{f^*k_1}{-(s^*+\varepsilon)}\,[1-e^{(s^*+\varepsilon)(t-t_0)}] \tag{A.8}$$

and therefore

$$|e_{\alpha k}(\delta)|\le\frac{2}{\rho_k\delta}\int_{t_0}^{t_0+\delta}|x_3(t)|e^{-\lambda t}|\sin\omega_k t|\,dt\le$$

$$\le\frac{2f^*k_1}{-(s^*+\varepsilon)\rho_k\delta}l\quad(k=\overline{1,n}),$$

where l is a bounded number because $\lambda>C_0\ge s^*+\varepsilon$.

So, the first relations (A.5) is proved. The proof of the second relation is analogously.