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LINEAR SYSTEMS

Design of Controllers by Indices of Precision and Speed. II. Nonminimal-Phase Plants

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Abstract—A method to design controllers of one-dimensional nonminimal-phase plants under unknown bounded external perturbations was proposed. It is based on determining the parameters of the Bézout identity. The attainable indices of precision and speed were determined.

Keywords: controller design, nonminimal-phase plants, control precision, robustness margins.

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1. INTRODUCTION

One of the main problems of automatic control is represented by the design of controllers in terms of the indices of precision, control time, and overshoot. The Bode plot is one of the basic problems of automatic control [1, 2]. It is a heuristic semigraphical method which hardly yields to computer-aided automation. In this connection, analytical methods of design in terms of these indices are worked up.

One of the lines of research relies on the linear-quadratic optimization [3, 4]. Numerous publications are concerned with the design of PI and PID controllers. Low order of the plant equations enables one to establish an explicit relation between the coefficients of such controllers and the times of control and overshoot, robustness margins, and so on. Several thousands of relations describing such couplings can be found in [5]. Design of the controllers for the minimal-phase plants satisfying the requirements on precision and speed under unknown bounded external perturbation is proposed in [6]. It is based on the relation of the roots of the characteristic system polynomial with these indices. If the controller polynomial is determined from the Bézout identity comprising a polynomial with certain roots, then the desired requirements can be satisfied by setting these roots.

A similar approach is used in the present paper to design the controller of a nonminimal-phase plant. However, for such plants the above relation is much more complicated. Additionally, it is common knowledge that for such plants no controller satisfying the requirements on precision and speed needs to exist.

In this connection, it is necessary to find the maximum permissible values of these indices. The maximum permissible precision can be established using [7, 8]. For example, in [7] this precision was obtained under the monofrequency harmonic external perturbation with unknown frequency, and in [8], for any bounded external perturbation.

[†] Deceased.

2. FORMULATION OF THE PROBLEM

Consider an asymptotically stable control system obeying the equations

$$y^{(n)} + d_{n-1}y^{(n-1)} + \ldots + d_1\dot{y} + d_0 y$$
$$= k_m u^{(m)} + \ldots + k_1\dot{u} + k_0 u + c_p f^{(p)} + \ldots + c_0 f, \quad m < n, \quad p < n, \tag{2.1}$$

$$g_{n_c} u^{(n_c)} + \ldots + f_1\dot{u} + g_0 u = r_{m_c} y^{(m_c)} + \ldots + r_1\dot{y} + r_0 y, \quad n_c \geqslant m_c, \tag{2.2}$$

where $y(t)$ is the measured output of plant (2.1) which is the controlled variable, $u(t)$ is the control generated by the controller (2.2), $f(t)$ is the unknown external perturbation which is bounded by a certain number f^* and represented by a polyharmonic function

$$f(t) = \sum_{i=1}^{N} f_i \sin(\omega_i t + \varphi_i) \tag{2.3}$$

with unknown frequencies ω_i and phases φ_i $(i = \overline{1, N})$, its unknown amplitudes being such that

$$\sum_{i=1}^{N} |f_i| \leqslant f^*.$$

If $f(t)$ is a sectionally continuous function, then (2.3) represents for $N \to \infty$ its expansion in the Fourier series.

Problem 1. For a certain and fully controllable plant (2.1), determine the controller (2.2) satisfying the requirements on precision

$$|y(t)| \leqslant y^*, \quad t \geqslant t_{\text{reg}}, \tag{2.4}$$

speed

$$t_{\text{reg}} \leqslant t^*_{\text{reg}}, \tag{2.5}$$

and margins of robustness

$$r_a \geqslant r^*_a, \tag{2.6}$$

where y^*, t^*_{reg}, and r^*_a are the given numbers.

The radius of robustness margins r_a is related with the phase (φ_{m}) and modulo (L) by

$$\varphi_{\text{m}} = 2\arccos\sqrt{1 - \frac{r_a^2}{4}}, \quad L = \min\left[1 + r_a,\ \frac{1}{1 - r_a}\right].$$

In particular, for $r_a = 0.75$, $\varphi_{\text{m}} = 42°$, $L = 1.75$. The radius of robustness margins can be established experimentally without opening the system.

The desired precision and speed should satisfy the conditions

$$y^* \geqslant y^{**}, \quad t^*_{\text{reg}} \geqslant t^{**}_{\text{reg}},$$

where the numbers y^{**} and t^{**}_{reg} are the permissible precision and speed. They are unknown, which gives rise to the following problem.

Problem 2. Needed is to determine for the given plant the permissible values of precision and speed.

3. ESSENCE OF THE APPROACH

By Laplace transforming Eqs. (2.1) and (2.2) under the zero conditions, we put down

$$d(s)y = k(s)u + c(s)f, \tag{3.1}$$

$$g(s)u = r(s)y, \tag{3.2}$$

$$d(s) = \sum_{i=0}^{n} d_i s^i, \quad k(s) = \sum_{i=0}^{m} k_i s^i, \quad g(s) = \sum_{i=0}^{n_c} g_i s^i,$$

$$r(s) = \sum_{i=0}^{m_c} r_i s^i, \quad c(s) = \sum_{i=0}^{p} c_i s^i.$$

Represent the polynomial $k(s)$ as

$$k(s) = k_1(s)k_2(s),$$

where $k_1(s)$ is a polynomial of degree m_1 whose roots have negative real parts, and $k_2(s)$ is a polynomial of degree m_2 $(m_1 + m_2 = m)$ whose roots have nonnegative real parts.

The plant (3.1) is nonminimal-phase if $m_2 \neq 0$ $(m_2 \leqslant m)$. The controller (3.2) is determined from the Bézout identity

$$d(s)g(s) - k(s)r(s) = \psi(s), \tag{3.3}$$

where $\psi(s)$ is the modal polynomial with roots having negative real parts.

In the left side of (3.3) there is the characteristic polynomial of system (3.1), (3.2), and in the right side, the desired characteristic polynomial. By comparing the coefficients at the identical degrees s, we get a system of linear algebraic equations for determination of the coefficients of the controller polynomials $g(s)$ and $r(s)$ providing the desired characteristic polynomial of system (3.1), (3.2).

Take the following structure of the modal polynomial

$$\psi(s) = k_1(s)\varepsilon(s)\delta_k(s)\delta(s), \tag{3.4}$$

where $\delta_k(s) = k_2(-s)$ and the realizability polynomial $\varepsilon(s)$ and the basic polynomial $\delta(s)$ are given by

$$\varepsilon(s) = \prod_{i=1}^{n-m} \left(\frac{\nu_i}{s_\delta} s + 1\right), \quad s_\delta = \max\left[s_{\delta,1}, \ldots, s_{\delta,n}\right], \tag{3.5}$$

$$\delta(s) = \prod_{i=1}^{n} \left(s + s_{\delta,i}\right),$$

where $s_{\delta,i},\ i = \overline{1,n}$, and $\nu_i,\ i = \overline{1, n-m}$, are the given positive numbers.

The desired polynomial

$$g(s) = k_1(s)g_\varepsilon(s)g_k(s)$$

has multipliers of the following degrees

$$\deg g_\varepsilon(s) = \deg \varepsilon(s), \quad \deg g_k(s) = \deg k_2(s).$$

By reducing the Bézout identity (3.3) by the polynomial $k_1(s)$, represent it as

$$d(s)g_\varepsilon(s)g_k(s) - k_2(s)r(s) = \varepsilon(s)\delta_k(s)\delta(s). \tag{3.6}$$

Denote $\tilde{d}(s) = d(s)g_k(s)$, $\tilde{r} = k_2(s)r(s)$, $\tilde{\delta}(s) = \delta_k(s)\delta(s)$ and represent this identity as

$$\tilde{d}(s)g_\varepsilon(s) - \tilde{r}(s) = \tilde{\delta}(s)\varepsilon(s).$$

It is easy to see that this identity coincides to within the notation with the corresponding identity [6]. Therefore, the following property is valid under a sufficiently small coefficients of the polynomial $\varepsilon(s)$—under small ν_i, $i = \overline{1, n-m}$.

Property 1. The coefficients of the polynomials $g_\varepsilon(s)$ and $\tilde{r}(s)$ are represented as:

$$g_{\varepsilon,i} = \varepsilon_i + 0_{1,i}(\nu), \quad i = \overline{0, n-m}, \quad r_j = \tilde{r}_j + 0_{2,j}(\nu), \quad j = \overline{0, n+m_2-1},$$

where $0_{1,i}(\nu)$ and $0_{2,j}(\nu)$ are the functions vanishing with the vector $\nu = [\nu_1, \ldots, \nu_{n-m}]$:

$$\lim_{\nu\to 0} 0_{1,i}(\nu) = 0, \quad \lim_{\nu\to 0} 0_{2,j}(\nu) = 0.$$

In connection with Property 1, we assume for simplicity in what follows that

$$g_\varepsilon(s) = \varepsilon(s). \tag{3.7}$$

The essence of the approach to satisfying the requirements on the system is based on the relation between the roots of the modal polynomial, the plant polynomial, and the denominator of the controller transfer function with the indices of precision, speed, and robustness. Similar to the relations described in [6], for $c(s) = c_0$ and condition (3.7), these relations are given by

$$\sup_{0\leqslant\omega<\infty} |t_{yf}(j\omega)| = \sup_{0\leqslant\omega<\infty} \frac{|g_k(j\omega)|\,|c_0|}{|\delta_k(j\omega)\delta(j\omega)|} \leqslant \frac{y^*}{f^*}, \tag{3.8}$$

$$s_{\delta,i} \geqslant \frac{\beta}{t^*_{\text{reg}}}, \quad i = \overline{1,n}, \quad \beta = 3, \tag{3.9}$$

$$r_a = \inf_{0\leqslant\omega<\infty} \frac{|\delta_k(j\omega)\delta(j\omega)|}{|g_k(j\omega)d(j\omega)|} \geqslant r_a^*. \tag{3.10}$$

In the case of minimal-phase plant, the polynomials $g_k(s)$ and $\delta_k(s)$ coincide in inequalities (3.8) and (3.10). Therefore, for satisfaction of these inequalities the roots of the basic polynomial are defined in [6] as

$$s_{\delta,i} = |s_{d,i}|\, q_t, \quad i = \overline{1,n}, \quad q_t > 1, \tag{3.11}$$

$$\prod_{i=1}^{n} s_{\delta,i} \leqslant \frac{f^*}{y^*}. \tag{3.12}$$

For the nonminimal-phase plant, the polynomial $g_k(s)$ depends intricately on the roots of the polynomials $\delta_k(s)$, $\delta(s)$, and $d(s)$; and now the problem lies in determining the roots of the polynomials $\delta_k(s)$ and $\delta(s)$ so that the inequalities (3.8)–(3.10) be satisfied.

4. SYSTEM PERFORMANCE INDICES VS. THE ROOTS OF THE POLYNOMIAL $k_2(s)$

4.1. Large Roots

Order the modules of the plant roots and those of the basic polynomial:

$$|s_{d,1}| \leqslant |s_{d,2}| \leqslant \ldots \leqslant |s_{d,n}|, \quad s_{\delta,1} \leqslant s_{\delta,2} \leqslant \ldots \leqslant s_{\delta,n}, \quad s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{m_2},$$

where s_i, $i = \overline{1, m_2}$, are the roots of the polynomial $k_2(s)$.

For simplicity, we confine our consideration to the real roots of the polynomial $k_2(s)$, examine the polynomial $g_k(s)$ under larger roots of the polynomial $k_2(s)$ as compared with the roots of the polynomial $d(s)$:

$$s_i > |s_{d,n}|\,\theta_d, \quad i = \overline{1, m_2}, \tag{4.1}$$

where θ_d is sufficiently larger positive number.

Select the roots of the polynomial $\delta(s)$ so that a similar condition be satisfied

$$s_i > s_{\delta,n}\theta_\delta, \quad i = \overline{1, m_2}. \tag{4.2}$$

Statement 1. *If inequalities* (4.1) *are satisfied for the control plant and the modules of the roots of the modal polynomial are selected from the condition* (4.2), *then for sufficiently large values of the numbers θ_d and θ_δ the polynomial of the controller*

$$g_k(s) = \delta_k(s) + 0(s, \theta_d, \theta_\delta), \tag{4.3}$$

where the polynomial $0(s, \theta_d, \theta_\delta)$ includes coefficients vanishing with growth of θ_d and θ_δ,

$$\lim_{\theta_d, \theta_\delta \to \infty} 0(s, \theta_d, \theta_\delta) = 0.$$

Proof. Consider the relations

$$\frac{\delta(s_i)}{d(s_i)} = \frac{\prod\limits_{p=1}^{n}(s_i + s_{\delta,p})}{\prod\limits_{p=1}^{n}(s_i + s_{d,p})} = \frac{\prod\limits_{p=1}^{n}\left(1 + \dfrac{s_{\delta,p}}{s_i}\right)\prod\limits_{i=1}^{n} s_i^n}{\prod\limits_{p=1}^{n}\left(1 + \dfrac{s_{d,p}}{s_i}\right)\prod\limits_{i=1}^{n} s_i^n}, \quad i = \overline{1, m_2}.$$

From inequalities (4.1) and (4.2), obtained are the relations

$$\frac{s_{\delta,p}}{s_i} < \frac{1}{\theta_\delta}, \quad \frac{|s_{d,p}|}{s_i} < \frac{1}{\theta_d}, \quad p = \overline{1, n}, \quad i = \overline{1, m_2}$$

from which it follows that

$$\frac{\delta(s_i)}{d(s_i)} = 1 + 0_i(\theta_d, \theta_\delta), \quad i = \overline{1, m_2}, \tag{4.4}$$

where the functions $0_i(\theta_d, \theta_\delta)$ feature $\lim_{\theta_d, \theta_\delta \to \infty} 0_i(\theta_d, \theta_\delta) = 0$, $i = \overline{1, m_2}$.

Taking into consideration (3.7) and (4.4) and disregarding the vanishing functions, set down the identity (3.6) as

$$\sum_{j=0}^{m_2} (s_i)^j\, g_{k,j} = \delta_k(s_i), \quad i = \overline{1, m_2}$$

from which Eq. (4.3) follows.

Consequently, under larger roots of the polynomial $k_2(s)$ closeness arises to the case of minimal-phase plant, and determination of the roots of the basic polynomial with the use of (3.9), (3.11) and (3.12) satisfies the requirements (2.4)–(2.6).

4.2. Smaller Roots

Consider the polynomial $g(s)$ under smaller roots of the polynomial $k_2(s)$ as compared with the roots $d(s)$:

$$s_i \leqslant \frac{|s_{d,1}|}{\underline{\theta}_d}, \quad i = \overline{1, m_2}, \tag{4.5}$$

where $\underline{\theta}_d$ is a sufficiently large positive number.

The roots of the basic polynomial are taken as follows:

$$s_i \leqslant \frac{s_{\delta,1}}{\underline{\theta}_\delta}, \quad i = \overline{1, m_2}, \tag{4.6}$$

where $\underline{\theta}_\delta$ is a sufficiently large positive number.

Statement 2. *If the roots of the polynomial $k_2(s)$ satisfy the inequalities* (4.5) *and* (4.6), *then for sufficiently large numbers $\underline{\theta}_d$ and $\underline{\theta}_\delta$ the controller polynomial is given by*

$$g_k(s) = \frac{\delta_0}{d_0}\delta_k(s) + 0\left(s,\, \underline{\theta}_d,\, \underline{\theta}_\delta\right), \tag{4.7}$$

where the polynomial $0\left(s,\, \underline{\theta}_d,\, \underline{\theta}_\delta\right)$ includes the coefficients vanishing with growing $\underline{\theta}_d$ and $\underline{\theta}_\delta$.

Proof. Consider the relations

$$\frac{\delta(s_i)}{d(s_i)} = \frac{\prod\limits_{p=1}^{n}(s_i + s_{\delta,p})}{\prod\limits_{p=1}^{n}(s_i + s_{d,p})} = \frac{\prod\limits_{p=1}^{n} s_{\delta,p}\left(1 + \dfrac{s_i}{s_{\delta,p}}\right)}{\prod\limits_{p=1}^{n} s_{d,p}\left(1 + \dfrac{s_i}{s_{d,p}}\right)} = \frac{\delta_0}{d_0}\left[1 + \underline{0}_i\left(\underline{\theta}_d, \underline{\theta}_\delta\right)\right], \quad i = \overline{1, m_2},$$

where $\underline{0}_i\left(\underline{\theta}_d, \underline{\theta}_\delta\right)$ is a function vanishing with growing numbers $\underline{\theta}_d$ and $\underline{\theta}_\delta$:

These relations follow from the inequalities

$$\frac{s_i}{s_{d,p}} < \frac{1}{\underline{\theta}_d}, \quad \frac{s_i}{s_{\delta,p}} < \frac{1}{\underline{\theta}_\delta}, \quad i = \overline{1, m_2}, \quad p = \overline{1, n}.$$

From the identity (3.6) we get that

$$\sum_{j=0}^{m_2} g_{k,j} s_i^j = \frac{\delta_0}{d_0} \sum_{j=0}^{m_2} \delta_j s_i^j, \quad i = \overline{1, m_2}. \tag{4.8}$$

Equation (4.7) follows from equalities (4.8).

The transfer function in (3.8) relating the system output with the external perturbation is now put down as

$$t_{yf}(s) = \frac{c_0 \delta_0}{\delta(s) d_0}.$$

This function for $s = 0$ (where it defines the system output under stepwise external perturbation) goes over

$$t_{yf}(0) = \frac{c_0}{d_0}.$$

This expression implies that the controller does not enhance precision of the asymptotically stable plants.

4.3. Design Procedure

An iterative procedure of design is proposed. Its first operations determine the permissible values of precision and speed.

Procedure.

Operation 1. Determine the roots of the basic polynomial from the desired speed

$$s_{\delta,i} = \left(\frac{\beta}{t^*_{\text{reg}}\, q_t}\right)\alpha^i, \quad i = \overline{1,n}, \quad q_t > 1, \tag{4.9}$$

where α is a positive number introduced to make aliquant the roots of the basic polynomial.

Along with the definition $\delta_k(s) = k_2(-s)$, it is sometimes convenient to determine the roots $s_{\delta_k,i}$, $i = \overline{1,m}$, of the polynomial $\delta_k(s)$, by selecting a companion to (4.9)

$$s_{\delta_k,i} = \left(\frac{\beta}{t^*_{\text{reg}}\, q_t}\right)\alpha^i, \quad i = \overline{1,m_2}, \quad q_t > 1.$$

If $s_{\delta,i} < |s_{d,i}|$, $i = \overline{1,n_1}$, $n_1 < n$, then we assume that

$$s_{\delta,i} = s_{d,i}, \quad i = \overline{1,n_1}.$$

Operation 2. Find the polynomial $g_k(s)$ by solving identity (3.6) under $\varepsilon(s) = 1$, and determine the radius of the robustness margin

$$r_a = \inf_{0\leqslant\omega<\infty} \frac{|\delta_k(j\omega)\,\delta(j\omega)|}{|g_k(j\omega)\,d(j\omega)|}.$$

If the robustness condition $r_a > r_a^*$ is satisfied, then go to Operation 3. Otherwise, return to Operation 1 and increase the number q_t until the robustness conditions are satisfied for $q_t = q_t^*$. Then, the permissible speed

$$t^{**}_{\text{reg}} = t^*_{\text{reg}} q_{t^{**}}.$$

Operation 3. Check the requirement on precision (3.8). If it is not satisfied, then generate the roots of the basic polynomial

$$s_{\delta,i} = |s_{d,i}|\, q_{t^{**}}\, q_y, \quad i = \overline{1,n}, \quad q_y > 1.$$

By solving the Bézout identity $\varepsilon(s) = 1$ determine the polynomial $g_k(s)$, verify satisfaction of requirement (3.8) and increase the number q_y until this identity is satisfied at retention of the robustness margins.

Operation 4. Solve the Bézout identity (3.3) with the modal polynomial (3.4) and generate the desired controller

$$k_1(s) g_k(s) g_\varepsilon(s) u = r(s) y.$$

5. INERTIAL CONTROLLER

5.1. Structure of the Bézout Identity for an Asymptotically Stable Plant

If plant (3.1) is asymptotically stable, then precision can be increased within the range of low frequencies of the external perturbation, and the system robustness retained using the inertial controller proposed in [3].

For that, we generate a "plant" by replacing in it the polynomial $d(s)$ by the polynomial $d(s)\rho(s)$, where the polynomial $\rho(s)$ comprises a sufficiently large time constant which after design is transferred to the controller. We assume for simplicity that

$$m_2 = m \quad (k_2(s) = k(s)).$$

The Bézout identity (3.6) with the basic polynomial $\delta(s) = d(s)$ is now given by

$$d(s)\rho(s)g_\varepsilon(s)g_k(s) - k(s)r(s) = \varepsilon(s)\delta_k(s)\delta_\rho(s)d(s), \tag{5.1}$$

where

$$\rho(s) = \sum_{i=0}^{n_2} \rho_i s^i, \quad \delta_\rho(s) = \sum_{i=0}^{n_2} \delta_{\rho,i} s^i, \quad n_2 > m, \quad \delta_k(s) = k(-s).$$

Assume the following structures of these polynomials

$$\rho(s) = k(-s)(\rho_1 s + 1), \quad \rho_1 > 0, \tag{5.2}$$

$$\delta_\rho(s) = \rho(s) + \mu k(s), \quad \mu > 0. \tag{5.3}$$

The numbers ρ_1 and μ are then determined from the conditions for system robustness and precision.

The degree of the desired polynomial $r(s)$ is given by

$$\deg [r(s)] = n + n_2 - 1.$$

Classify the introduced polynomial ρ with the controller. Then, it becomes

$$\rho(s)g_\varepsilon(s)g_k(s)u = r(s)y, \tag{5.4}$$

where the degree of polynomial $g_\varepsilon(s)$ is given by

$$\deg [g_\varepsilon(s)] = \deg [\varepsilon(s)] = n - m - 1.$$

5.2. Properties of the Solutions of the Bézout Identity for Plants under $m = n - 1$

There is no difficulty in seeing that the controller (5.4) is realizable if the realizability polynomial $\varepsilon(s) = 1$ and, consequently, $g_\varepsilon(s) = 1$, provided that the plant has degree $m = n - 1$.

The identity (5.1) takes on the form

$$d(s)\rho(s)g_k(s) - k(s)r(s) = \delta_k(s)\delta_\rho(s)d(s). \tag{5.5}$$

Property 2. The desired controller polynomial $g_k(s)$ is given by

$$g_k(s) = \delta_k(s). \tag{5.6}$$

Indeed, we establish from the identity (5.5) for $s = s_i$ that

$$g_k(s_i) = \frac{\delta_k(s_i)\delta_\rho(s_i)}{\rho(s_i)}, \quad i = \overline{1, m}. \tag{5.7}$$

It follows from (5.3) that

$$\delta_\rho(s_i) = \rho(s_i), \quad i = \overline{1, m}.$$

Then, from equalities (5.7) we get the relations

$$g_k(s_i) = \delta_k(s_i), \quad i = \overline{1, m}$$

providing property (5.6).

5.3. Determination of the Numbers μ and ρ_1

First, determine the numbers μ and ρ_1 from the condition for negativeness of the real parts of the roots of the polynomial $\delta_\rho(s)$.

Order the time constants of the polynomial

$$k(s) = k_0 \prod_{i=1}^{m} (-T_i s + 1), \quad T_i > 0, \quad i = \overline{1, m}, \quad k_0 > 0,$$

as

$$T_1 > T_2 > \ldots > T_m,$$

and introduce the parameter θ taking on values $\theta = \overline{5, 10}$.

If $T_2, \ldots, T_m$ are sufficiently small as compared with T_1, then $\theta = 1$.

Statement 3. *The roots of the polynomial $\delta_\rho(s)$ have negative real parts if*

$$\mu = \frac{\sqrt{\rho_1^2 + T_1^2\theta^2}}{T_1\theta}. \tag{5.8}$$

Proof. The proof uses the Nyquist plot. In this connection, we consider the amplitude–frequency $a(\omega)$ and phase–frequency $\varphi(\omega)$ characteristics corresponding to the transfer function $w_\rho(s)$ obeying the relation

$$\frac{\delta_\rho(s)}{\rho(s)} = 1 + w_\rho(s),$$

where

$$\mathrm{w}_\rho(s) = \mu \frac{k(s)}{k(-s)(\rho_1 s + 1)}$$

with regard for (5.2), (5.3).

These characteristics are given by

$$a(\omega) = \frac{\mu}{\sqrt{\rho_1^2\omega^2 + 1}}, \quad 0 \leqslant \omega < \infty, \tag{5.9}$$

$$\varphi(\omega) = -\arctan \rho_1\omega - 2\sum_{i=1}^{m} \arctan T_i\omega, \quad 0 \leqslant \omega < \infty.$$

The system's crossover frequency is determined from (5.9) as

$$\omega_{\mathrm{cr}}^2 = \frac{\mu^2 - 1}{\rho_1^2}. \tag{5.10}$$

Determine the numbers μ and ρ_1 so that the system crossover frequency be essentially to the left of the frequency corresponding to the greatest time constant of the polynomial $k(s)$:

$$\omega_{\mathrm{cr}} = \frac{1}{T_1\theta}.$$

By using (5.10) one can readily see that this is provided by relation (5.8).

Now, determine the number ρ_1 for which requirement (2.5) on precision is satisfied under sufficiently small frequencies of external perturbation and robustness.

In this connection, we put down the system transfer functions and the radius of robustness margins as

$$t_{yf}(s) = \frac{\rho(s)}{\delta_\rho(s)d(s)}, \quad r_a = \inf_{0\leqslant\omega<\infty} \frac{|\delta_\rho(j\omega)|}{|\rho(j\omega)|}. \tag{5.11}$$

Under a staircase external perturbation ($f_{\text{step}} = f^*$ for $t \geqslant t_0$ and $f_{\text{step}} = 0$ for $t < t_0$) the system output is defined by

$$|t_{yf}(0)| = \frac{|c_0|\,|k_0|}{|d_0|\,|k_0 + k_0\mu|} = \frac{|c_0|}{|d_0|\,|1+\mu|}.$$

Using (5.8), we put down the condition for precision

$$\frac{|c_0|\,T_1\theta}{|d_0|\left|T_1\theta + \sqrt{\rho_1^2 + T_1^2\theta^2}\right|} \leqslant \frac{f^*}{y^*}, \tag{5.12}$$

from which the parameter ρ_1 is determined, and then from (5.8) determine the number μ and verify on the basis of (5.11) the robustness condition $r_a \geqslant r_a^*$.

5.4. Plants for $m < n-1$

If the degree of the polynomial $k(s)$ of plant (3.1) is smaller than $n-1$, then generate the realizability polynomial (3.5) of degree $n-m-1$ and solve the Bézout identity (5.1) using the values of μ and ρ_1 as determined above.

6. TRACKING SYSTEM

The tracking system obeys the equations

$$\begin{aligned} d(s)y &= k(s)u + c(s)f, \\ d(s)u &= r(s)y + r_{\text{cr}}(s)y_r, \end{aligned} \tag{6.1}$$

where $y_r(t)$ is the measured reference action which is a sectionally constant function with sufficiently large constancy intervals.

The problem lies in determining the controller polynomials (6.1) under which satisfied are the requirements on precision

$$|e(t)| \leqslant e^*, \quad t \geqslant t_{\text{reg}},$$

where $e(t)$ are the deviations of the plant output from the reference action

$$e(t) = y(t) - y_r(t),$$

e^* is the given number (permissible tracking error), as well as speed (2.5) and robustness margin (2.6).

The plant output is related with the external perturbation for $c(s) = c_0$ and reference action in the form of

$$y = \frac{g(s)c_0}{d(s)g(s) - k(s)r(s)}f + \frac{k(s)r_{\text{cr}}(s)}{d(s)g(s) - k(s)r(s)}y_r.$$

Further, we assume that the polynomials $g(s)$ and $r(s)$ of controller (6.1) were obtained by the procedure (see p. 967). Then, we obtain under the condition (3.7) that

$$y = \frac{g_k(s)c_0}{\delta_k(s)\delta(s)} f + \frac{k_2(s)r_{\mathrm{cr}}(s)}{\delta_k(s)\delta(s)}\, y_r. \tag{6.2}$$

Represent the basic polynomial as

$$\delta(s) = \delta_1(s)\delta_2(s),$$

where

$$\delta_1(s) = \prod_{i=1}^{n_1}(s + s_{\delta,i}), \quad \delta_2(s) = \prod_{i=n_1+1}^{n}(s + s_{\delta,i}), \quad n_1 \leqslant n,$$

and denote

$$a_1 = \prod_{i=1}^{n_1} s_{\delta,i}.$$

For $\delta_k(s) = k_2(-s)$ take the controller polynomial $r_{\mathrm{cr}}(s)$ as

$$r_{\mathrm{cr}}(s) = a_1\delta_2(s)$$

and represent (6.2) as

$$y = y_f + y_y,$$

where

$$y_f = \frac{g_k(s)c_0}{k_2(-s)\delta(s)} f, \quad y_y = \frac{k_2(s)a_1}{k_2(-s)\delta_1(s)}\, y_r.$$

It follows from the above relations that

$$|y_f(t)| \leqslant y^*, \quad y_y(t) = y_r \quad \text{for} \quad t \geqslant t_{\mathrm{reg}}.$$

7. EXAMPLE

7.1. Use of the Procedure

Consider the plant obeying the equation

$$\dddot{y} + 6.25\ddot{y} + 26.2\dot{y} + 5y = -2\dot{u} + 5u + 5f,$$

where $f(t)$ is a polyharmonic function (2.3) with the constraint $f^* = 1$.

It is desired to determine a controller providing precision

$$|y(t)| \leqslant 0.1, \quad t \geqslant t_{\mathrm{reg}},$$

speed

$$t_{\mathrm{reg}} < 0.3 \tag{7.1}$$

and robustness margins in phase and modulo.

To design the controller, we first use the procedure.

According to Operation 1 of this procedure, we generate the roots of the basic polynomial and the polynomial $\delta_k(s)$:

$$s_{\delta,1} = 8.46, \quad s_{\delta,2} = 9.3, \quad s_{\delta,3} = 10.23, \quad s_{\delta_k,1} = 10$$

on the basis of requirements on speed (7.1) and robustness margins (the roots of the plant polynomial: $|s_{d,1}| = 0.2$, $|s_{d,2}| = 5$, $|s_{d,3}| = 5$.)

To execute Operation 2, we generate the Bézout identity

$$(s^3 + 6.25s^2 + 26.2s + 5)(g_{k1}s + g_{k0}) - (-2s + 5)(r_2s^2 + r_1s + r_0)$$
$$= (s + 10)(s + 8.46)(s + 9.3)(s + 10.23),$$

solve it and obtain the polynomial $g_k(s) = (s + 162.5)$. Determine the radius of the robustness margins

$$r_a = 0.18,$$

which does not provide system robustness. In this connection, the number q_t is increased, the operation is repeated anew, and the new radius of robustness margins is determined. Four iterations provide the radius of robustness margins $r_a = 0.69$. At that, the permissible control time $t_{\text{reg}} = 5$ c.

To realize the controller, the realizability polynomial $\varepsilon(s) = 0.179s + 1$ is added to the right side.

In this case, solution of the Bézout identity provides a controller with the transfer function

$$w_c(s) = -\frac{1.749s^2 + 11.15s + 67.5}{0.179s^2 + 3.185s + 20.12}.$$

According to Operation 4 of the procedure, we find

$$\sup_{0 \leqslant \omega < \infty} |t_{yf}(j\omega)| \leqslant 0.77.$$

This permissible precision differs from the desired one almost by the factor of ten.

7.2. Inertial Controller

Generate a modal polynomial of identity (5.1)

$$\psi(s) = (\varepsilon_1 s + 1)(2s + 5)\left[(2s + 3)(\rho_1 s + 1) + \mu(-2s + 5)\right](s^3 + 6.25s^2 + 26.2s + 5).$$

Using inequality (5.12), determine the number ρ_1. Find from (5.8) the parameter μ

$$\rho_1 = 20, \quad \mu = 10$$

and take $\varepsilon_1 = 0.1$.

By solving the Bézout identity (5.1), obtain a controller with the transfer function

$$W_c(s) = -\frac{3.94s^4 + 34.3s^3 + 64s^2 + 278s + 49.2}{4.549s^4 + 36.6s^3 + 109s^2 + 119s + 5.7}$$

providing under the staircase external perturbation the desired precision

$$|y(t)| \leqslant 0.1, \quad t \geqslant t_{\text{reg}},$$

control time

$$t_{\text{reg}} = 20\,\text{s},$$

and radius of the robustness margins

$$r_a = 0.7.$$

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