FREQUENCY PROPERTIES OF OPTIMAL LINEAR SYSTEMS WITH SEVERAL CONTROLS

A. G. Aleksandrov

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The optimality conditions are derived in frequency form for systems with m controls. The definitions of modulus and phase stability margins (L) and (φ^*) are introduced, as well as the definitions of oscillatability index (M) and cutoff frequency. As a result of the analysis of the derived optimality conditions with consideration of the definitions which have been introduced, it is shown that, just as in the case m = 1 [1], the frequency properties of optimal systems are described by the relationships $\varphi^* \geq 60°$, $L \geq 2$, and $M \leq 2$.

1. Definitions and Notation

Assume that we have a control system whose asymptotically stable perturbed motion can be described by the equations

$$\dot{x} = Px + Bu, \tag{1.1}$$

$$u = C'x, \tag{1.2}$$

where P, B, C are number matrices having the respective dimensionalities $n \times n$, $n \times m$, and $m \times n$; x(t) is the n-dimensional vector representing the phase coordinates of the plant; u(t) is the m-dimensional control vector.

We shall assume that the plant is fully controllable and that the control law (1.2) is fully observable.

Going over in (1.1), (1.2) to Laplace transforms for zero initial conditions, we formulate the transfer matrices W(s) and $W_{clo}(s)$ of this system in the open-loop and closed-loop states [2]:

$$W(s) = -C'(Es - P)^{-1}B, \tag{1.3}$$

$$W_{clo}(s) = -\{E + W(s)\}^{-1}W(s). \tag{1.4}$$

The elements $w_{ij}(s)$, $w_{ij\,clo}(s)$ $(i, j = 1, \ldots, n)$ of these transfer matrices are the transfer functions of the system (1.1), (1.2) from the j-th coordinate of the input vector to the i-th coordinate of the output vector.

The transfer matrices (1.3), (1.2) can be derived from the equations

$$sx = Px + B(u + r), \tag{1.1'}$$

$$u = C'x, \tag{1.2'}$$

where r(s) is the m-dimensional perturbation vector.

In fact, if we place $u \equiv 0$ in (1.1') (this corresponds to opening of the system), then from (1.1'), (1.2') it follows that

$$u = C'x = C'(Es - P)^{-1}Br = -W(s)r;$$

however, if the value of the vector u in (1.1') is determined on the basis of (1.2'), then we obtain

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$$u = C'x = C'(Es - P)^{-1}B(u + r),$$

whence

$$u = \{E - C'(Es - P)^{-1}B\}^{-1}C'(Es - P)^{-1}Br = -\{E + W(s)\}^{-1}W(s)r.$$

The characteristic equations of the system (1.1), (1.2) in the open-loop and closed-loop states have the form

$$D_{clo}(s) = \begin{vmatrix} Es - P & B \\ C' & E \end{vmatrix} = 0, \tag{1.5}$$

$$D_0(s) = |Es - P| = 0. \tag{1.6}$$

Using the properties of determinants comprised of four blocks [3], we represent the characteristic polynomial of Eq. (1.5) in the following form for the condition $D_0(s) \neq 0$:

$$D_{clo}(s) = |Es - P - BC'| = |Es - P|\ |E - C'(Es - P)^{-1}B| = D_0(s)\,|E + W(s)|. \tag{1.7}$$

We introduce the definitions of the cutoff frequency, the stability margin, and the oscillatability index for the system (1.1), (1.2). Note first that the mathematical expression for these characteristics in the case of systems with one control can be represented in the form of relationships between the characteristic polynomials $D_{clo}(s)$ and $D_0(s)$ for various values of $s = j\omega$. This relationship, in general, does not depend on the control-system structures corresponding to the specific polynomials $D_{clo}(s)$ and $D_0(s)$.

In fact, if we use the substitution

$$\psi_1(s) = D_{clo}(s) / D_0(s), \tag{1.8}$$

the stability of the control system is determined on the basis of the Nyquist-Mikhailov criterion [4, 5] by the shape of the curve

$$\psi_2(j\omega) = \psi_1(j\omega) - 1 = \frac{D_{clo}(j\omega)}{D_0(j\omega)} - 1 \tag{1.9}$$

on the $\operatorname{Im}\psi_2(j\omega)$, $\operatorname{Re}\psi_2(j\omega)$ plane. The phase and amplitude stability margins and the oscillatability index are determined from the following expressions, respectively:

$$\varphi^* = \pi + \arg \psi_2(j\omega_c), \tag{1.10}$$

$$L = \min\left\{\operatorname{Re}\psi_2(j\omega_1),\ \frac{1}{\operatorname{Re}\psi_2(j\omega_2)}\right\}, \tag{1.11}$$

$$M = \max_{0 \leqslant \omega \leqslant \infty} \frac{\operatorname{mod}\psi_2(j\omega)}{\operatorname{mod}\psi_1(j\omega)}, \tag{1.12}$$

where ω_1 and ω_2 satisfy the condition

$$\operatorname{Im}\psi_2(j\omega) = 0, \tag{1.13}$$

and the cutoff frequency ω_c is determined from the equation

$$\sqrt{\psi_2(j\omega_c)\psi_2(-j\omega_c)} = 1. \tag{1.14}$$

It is natural to determine the cutoff frequency, the stability margin, and the oscillatability index for systems having several controls on the basis of Eqs. (1.10)-(1.14) (it is obvious that for systems with one control the function $\psi_2(s)$ is the transfer function of the open-loop system).

Note that in determining the stability margin of the system (1.1), (1.2) by means of Eqs. (1.10), (1.11) it is assumed, just as in the case with one control [4, 5], that ω_c is unique and that the real part of the roots of the polynomial $D_0(s)$ is nonpositive.

2. The Frequency Form of the Optimality Condition for Systems with Several Controls

We shall assume hereafter that the system described by the equations

$$\dot{x} = Px + Bu, \tag{2.1}$$

$$u = C'x, \tag{2.2}$$

is optimal in the sense of the positive definite functional

$$J = \int_0^\infty (x'Qx + u'u)\,dt, \tag{2.3}$$

i.e., the controls (2.2) are such that the function (2.3) having the positive definite number matrix Q is minimized along the solutions of (2.1), (2.2).

The optimality of the control (2.2) means that the matrix C satisfies the following algebraic equations jointly with a certain positive definite matrix A [6]:

$$-AP - P'A + ABB'A = Q, \tag{2.4}$$

$$AB = -C. \tag{2.5}$$

In order to obtain the optimality conditions in frequency form we add and subtract the quantity sA in the left side of (2.4) and multiply, as in [7], the resulting equations by $B'(-Es-P)^{-1'}$ on the left side and by $(Es-P)^{-1}B$ on the right side; as a result, we obtain

$$\begin{gathered} B'(-Es-P)^{-1'}AB + B'A(Es-P)^{-1}B \\ + B'(-Es-P)^{-1'}(-Q + ABB'A)(Es-P)^{-1}B = 0. \end{gathered} \tag{2.6}$$

Representing the matrix Q in the form Q = H'H (Q and H are number matrices having the dimensionality $n \times n$) and introducing the substitution

$$H(s) = H(Es-P)^{-1}B, \tag{2.7}$$

we obtain the following result from (2.6) while taking account of (2.5), (2.7), and (1.3):

$$W'(-s) + W(s) + W'(-s)W(s) = H'(-s)H(s). \tag{2.8}$$

Adding a unit matrix to both sides of Eq. (2.8), we finally obtain

$$\{E + W(-s)\}'\{E + W(s)\} = E + H'(-s)H(s). \tag{2.9}$$

Equation (2.9) is the optimality condition for the control laws (2.2) for $s = j\omega$, expressed in frequency form.

Based on (2.9), it is similarly possible to write

$$|E + W(-s)|\,|E + W(s)| = |E + H'(-s)H(s)|. \tag{2.10}$$

3. The Frequency Properties of the System (2.1), (2.2), Which Do Not Depend on the Specific Choice of the Parameters of the Optimization Functional

We first express the function $\psi_2(s)$ appearing in Eqs. (1.10)-(1.14) in terms of the elements of the transfer matrix of the open-loop system (2.1), (2.2).

Based on (1.7), (1.8), we can write

$$\psi_1(s) = |E + W(s)|. \tag{3.1}$$

Using the properties of the determinant of the sum of the matrices [8], we represent this determinant in the form

$$|E + W(s)| = 1 + R(s), \tag{3.2}$$

Thus, if the matrix Q of the coefficients of the optimization functional (1.3) is chosen in such a way that the equation

$$|E+H'(-i\omega_c^*)H(j\omega_c^*)|=2 \tag{4.3}$$

is satisfied, it follows that the value of ω_c^* differs from unity by no more than a factor of 2.5 at the frequency $N(\omega_c^*)$.

Let us now examine the relationship between the elements of the matrix Q and the error vector. Under these conditions we restrict ourselves to the case in which not all of the phase coordinates of the plant (2.2) are accessible to measurement, but only the coordinates of a certain m-dimensional vector y can be measured; this vector is related to the vector x by the relationship

$$y=Dx, \tag{4.4}$$

where D is a number matrix having the dimensionality m x n.

Assuming complete observability of the vector x, we express the latter in terms of the coordinates of the measured vector y and represent the controls (2.2) in the form

$$u=C'(Es-P)^{-1}B\{D(Es-P)^{-1}B\}^{-1}y. \tag{4.5}$$

Equation (4.5) is not difficult to derive after eliminating u from the expressions

$$x=(Es-P)^{-1}Bu, \qquad y=D(Es-P)^{-1}Bu$$

and substituting the result into (2.2).

We establish a relationship between the perturbation vector (r) and the error vector (y) which is produced by this perturbation in a system having

$$sx=Px+Bu+Mr, \tag{4.6}$$

where M is a number matrix having the dimensionality n x m, and the controller equation is given by (4.5).

Equation (4.6) can be represented in the following form while taking account of (4.4):

$$\{D(Es-P)^{-1}B\}^{-1}y=u+\{D(Es-P)^{-1}B\}^{-1}\{D(Es-P)^{-1}M\}r,$$

whence it follows with allowance for (4.5) that

$$\{E+W(s)\}\{D(Es-P)^{-1}B\}^{-1}y=\{D(Es-P)^{-1}B\}^{-1}\{D(Es-P)M\}r \tag{4.7}$$

or

$$\begin{aligned} y'(-s)\{D(-Es-P)^{-1}B\}^{-1'}\{E+W(-s)\}'\{E+W(s)\}\ \{D(Es-P)^{-1}B\}^{-1}y(s) \\ =r'(-s)\{D(-Es-P)^{-1}M\}'\{D(-Es-P)^{-1}B\}^{-1'} \\ \times\{D(Es-P)^{-1}B\}^{-1}\{D(Es-P)^{-1}M\}r(s). \end{aligned} \tag{4.8}$$

Taking account of (2.9), we write (4.8) in the form

$$\begin{aligned} y'(-s)\{D(-Es-P)^{-1}B\}^{-1'}\{E+H'(-s)H(s)\}\{D(Es-P)^{-1}B\}^{-1}y(s) \\ =r'(-s)\{D(-Es-P)^{-1}M\}'\{D(-Es-P)^{-1}B\}^{-1'} \\ \{D(Es-P)^{-1}B\}^{-1}\{D(Es-P)^{-1}M\}r(s). \end{aligned} \tag{4.9}$$

If r(t) is a vector-function of the form

$$r(t)=\begin{cases}0 & \text{for} \quad t\leqslant 0, \\ r=\text{const} & \text{for} \quad t>0,\end{cases} \tag{4.10}$$

then it follows that for $t\to\infty$ it is possible to write the following equation on the basis of (4.7) while taking account of (2.9) for s = 0:

$$y'_{st}\{D\tilde{P}B\}^{-1'}\{ED_p^2(0)+B'\tilde{P}'Q\tilde{P}B\}\{D\tilde{P}B\}^{-1}y_{st}=r'\{D\tilde{P}M\}'\{D\tilde{P}B\}^{-1'}\{D\tilde{P}B\}^{-1}\{D\tilde{P}M\}r, \tag{4.11}$$

where $\tilde{P}$ is the matrix that is the reciprocal [3] of the matrix P; i.e.,

$$\frac{\tilde{P}}{|P|} = P^{-1}.$$

Equation (4.11) is a constraint on the choice of the matrix Q of the optimized functional for stipulated values of the error vector (y_{st}) which is caused by the stipulated perturbation (4.10) in the steady-state mode.

Note that if the functional (2.3) has the form

$$J = \int_0^\infty (y'Q_1y + u'u)\,dt, \tag{4.12}$$

it follows that for astatic systems the condition (4.11) takes the form

$$y_{st}'Q_1y_{st} = r'\{DPM\}'\{DPB\}^{-1'}\{DPB\}^{-1}\{DPM\}r. \tag{4.13}$$

(For the condition $w_{ii}(0) \gg 1$, Eq. (4.13) is similarly valid for the systems with a zero-order astatism.)

If M = B, it follows from (4.13) that

$$y_{st}'Qy_{st} = r'r. \tag{4.14}$$

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