DISCRETE SYSTEMS

CONSTRUCTION OF DISCRETE CONTROL SYSTEMS WITH GIVEN PROPERTIES

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UDC 62-503.4

We investigate the properties of a discrete control system, on the motions of which a positive quadratic functional is minimized. A relationship is established between the parameters of this functional and the frequency and accuracy characteristics of an optimal system. Controllers are constructed that ensure that the system will have given frequency properties and transfer constants.

1. Formulation of the Problem

We consider a discrete control system, whose asymptotically stable perturbed motion is described by the equations

$$x[(k+1)T] = \Phi x(kT) + Ru(kT) \quad (k = 0, 1, \ldots), \tag{1.1}$$

$$u(kT) = C'x(kT) \quad (k = 0, 1, \ldots), \tag{1.2}$$

where $x(kT)$ is an n-dimensional vector of the phase coordinates of the plant; $u(kT)$ is an m-dimensional vector of the output coordinates of the controllers, described by Eq. (1.2) (the vector u will also be called a vector of the inputs of the plant); Φ and R are known matrices of numbers of dimensions $n \times n$ and $n \times m$, respectively, satisfying the conditions of complete controllability; C' is an $m \times n$ matrix (the prime is the symbol of transposition); and T is a given interval of discreteness. For definiteness, we will assume that system (1.1), (1.2) contains m controllers each of which is described by the corresponding row of the matrix equation (1.2).

The margin of stability and the dynamic properties of system (1.1), (1.2) will be estimated using frequency performance indices [1]: the phase margin φ_M, the modulus margin L, and the multidimensional system (1.1), (1.2).

We break the system (1.1), (1.2) at the ν-th input of the plant, and we feed an action on this input in the form of a harmonic lattice function $\psi_\nu = -1 \sin \omega kT$ $(0 \leq \omega \leq \pi/T,\ k = 0, 1, \ldots)$. The reaction excited by this action on the output of the ν-th controller has the form

$$u_\nu = A_\nu(\omega) \sin[\omega kT + \varphi_\nu(\omega)]. \tag{1.3}$$

The block diagram of system (1.1), (1.2), broken at the ν-th input of the plant, is shown in Fig. 1.

We use the following notation: the symbol ν above a matrix denotes the matrix obtained by deleting the ν-th column. $\overset{\nu}{R}$ is a matrix of dimension $n \times (m-1)$, consisting of $m-1$ columns of the matrix R, u is an $(m-1)$-dimensional vector with components $\{u_1, \ldots, u_{\nu-1}, u_{\nu+1}, \ldots, u_m\}$.

It is known that $A_\nu(\omega)$ is a periodic function of period $2\pi/T$; therefore, as usual, we introduce a pseudofrequency

$$v = \operatorname{tg}(\omega T/2), \tag{1.4}$$

and then as ω varies from 0 to π/T, the pseudofrequency v varies from 0 to ∞.

Saratov. Translated from Avtomatika i Telemekhanika, No. 9, pp. 57-66, September, 1973. Original article submitted July 11, 1972.

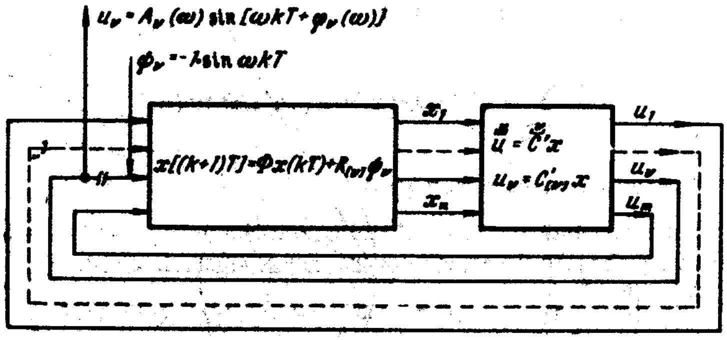

Fig. 1

Converting in the functions $A_\nu(\omega)$ and $\varphi_\nu(\omega)$ to the pseudofrequency v, we obtain the functions $A_\nu(v)$ and $\varphi_\nu(v)$, which are called the amplitude–frequency and phase–frequency characteristics of system (1.1), (1.2), broken at the ν-th input of the plant.

Repeating the procedure described for each of the m inputs of the plant, we obtain 2m frequency characteristics $A_i(v)$, $\varphi_i(v)$ ($i = 1, \ldots, m$); by using these characteristics we determine the stability margin of the multidimensional system (1.1), (1.2):

$$\varphi_M = \min\{\varphi_{M^1}, \ldots, \varphi_{M^m}\}, \quad L = \min\{L_1, \ldots, L_m\}, \quad M = \max\{M_1, \ldots, M_m\}, \tag{1.5}$$

where the φ_M, L_i, M_i ($i = 1, \ldots, m$) are defined by known equations [1, 2].

System (1.1), (1.2) is assumed to be "good" with respect to its frequency performance indices, if the values of the numbers φ, L, and M occur within the limits

$$\varphi_s = 30° - 60°, \quad L = 2 - 10, \quad M = 1.3 - 2. \tag{1.6}$$

Carrying out a z transformation on system (1.1), (1.2) for zero initial conditions, we write the expression for the transfer matrix $W_{br}(z)$ of this system in the broken state:

$$W_{br}(z) = -C'(Ez - \Phi)^{-1}R. \tag{1.7}$$

The matrix $W_{br}(1)$ is called the matrix of the transfer constants of the system (1.1), (1.2) (for m = 1 the scalar $k_p = w(1)$ is called the transfer constant of the system). It characterizes the values of the static errors of the system for a constant external action.

Larger values of the elements of the matrix $W_{br}(1) = \|w_{ij}(1)\|_1^m$ correspond to higher static accuracy of the system.

Below we consider the following problem: to determine a matrix C' such that the system (1.1), (1.2) has performance indices occurring within the limits (1.6), and the values of the transfer constants $w_{ij}(1)$ ($i, j = 1, \ldots, m$) satisfy the inequalities

$$\sum_{j=1}^{m} w_{ij}^2(1) \geq k_i^{*2} \quad (i = 1, \ldots, m), \tag{1.8}$$

where the k_i^* are given numbers.

For m = 1, relations (1.8) can be written in the form $k_p \geq k_p^*$ (k_p^* is a given number).

2. Frequency Performance Indices of Optimal Discrete Systems

We next assume that the matrix C' in Eq. (1.2) is obtained as a result of solving the problem of the analytical construction of controllers of discrete systems [3, 4]. This problem is formulated as follows.

Problem 2.1. Let there be a control plant, described by Eq. (1.1). It is required to determine the matrix C' of the equation of the controllers (1.2) such that on asymptotically stable motions of system (1.1), (1.2), excited by arbitrary initial deviations, the functional

$$I=\sum_{k=0}^{\infty}x'(kT)Qx(kT)+\mu_0^2u'(kT)u(kT) \tag{2.1}$$

is minimized, in which Q is a positive definite matrix.

An analytical and numerical solution of this problem is well known [3, 4] and reduces, in particular, according to the Lyapunov–Bellman method, to the solution of the matrix algebraic equation

$$(Q+A)-\Phi'(Q+A)\Phi+\Phi'(Q+A)R[R'(Q+A)R+E\mu_0^2]^{-1}R'(Q+A)\Phi=Q, \tag{2.2}$$

where A is an $n \times n$ positive definite matrix, E is the unit matrix, and it reduces to the successive calculation of the sought matrix C according to the equation

$$C'=-[R'(Q+A)R+E\mu_0^2]^{-1}R'(Q+A)\Phi. \tag{2.3}$$

Equations (2.2), (2.3) are called the algebraic conditions of the optimality of the system (1.1), (1.2). On the basis of these equations it is not difficult to obtain an optimality expression for the system under consideration in the frequency form.

Actually, adding and calculating from the left side of Eq. (2.2) the expression $\Phi'(Q+A)z$, we obtain after multiplying this equation on the left by $R'(Ez^{-1}-\Phi')^{-1}$ and on the right by $(Ez-\Phi)^{-1}R$:

$$R'(Q+A)z(Ez-\Phi)^{-1}R+R'(Ez^{-1}-\Phi')^{-1}\Phi'(Q+A)R+$$
$$+R'(Ez^{-1}-\Phi')^{-1}\Phi'(Q+A)R[R'(Q+A)R+E\mu_0^2]\times$$
$$\times R'(Q+A)\Phi(Ez-\Phi)^{-1}R=R'(Ez^{-1}-\Phi')^{-1}Q(Ez-\Phi)^{-1}R.$$

On the basis of (1.7), (2.3) we can write

$$W_{br}(z)=[R'(Q+A)R+E\mu_0^2]^{-1}R'(Q+A)\Phi(Ez-\Phi)^{-1}R. \tag{2.4}$$

Introducing the notation

$$H(z)=H(Ez-\Phi)^{-1}R,\quad R'(Q+A)R+E\mu_0^2=N,\quad H'H=Q \tag{2.5}$$

and using the identity $z(Ez-\Phi)^{-1}=E+\Phi(Ez-\Phi)^{-1}$, we obtain

$$N+NW_{br}(z)+W_{br}(z^{-1})N+W'_{br}(z^{-1})NW_{br}(z)=$$
$$=E\mu_0^2+H'(z^{-1})H(z). \tag{2.6}$$

Carrying out a w transformation ($z=(1+w)/(1-w)$) in (1.8), and replacing w by jv, we obtain the optimality condition for system (1.1), (1.2) in the frequency form

$$N+NW_{br}(jv)+W'_{br}(-jv)N+W'_{br}(-jv)NW_{br}(jv)=$$
$$=E\mu_0^2+H'(-jv)H(jv). \tag{2.7}$$

For the case of a single controller ($m = 1$) the identity (2.7) takes the form [5]

$$[1+w_{br}(-jv)][1+w_{br}(jv)]=$$
$$=(1-\varepsilon)\left[1+\mu_0^{-2}\sum_{i=1}^{n}h_i(-jv)h_i(jv)\right], \tag{2.8}$$

where

$$\varepsilon=\frac{r'(Q+A)r}{\mu_0^2+r'(Q+A)r}, \tag{2.9}$$

$n_i(jv)$ ($i = 1, \ldots, n$) are the components of the vector $H(z) = H(Ez-\Phi)^{-1}r$, $z = z=(1+jv)/(1-jv)$, $r = R$ for $m = 1$.

Equation (2.9) for the coefficient ε contains the matrix Q + A, characterizing the value I_{opt} of the functional (2.1) on the extrema (1.1), (1.2). We also have $I_{opt} = x'(0)(Q+A)x(0)$.

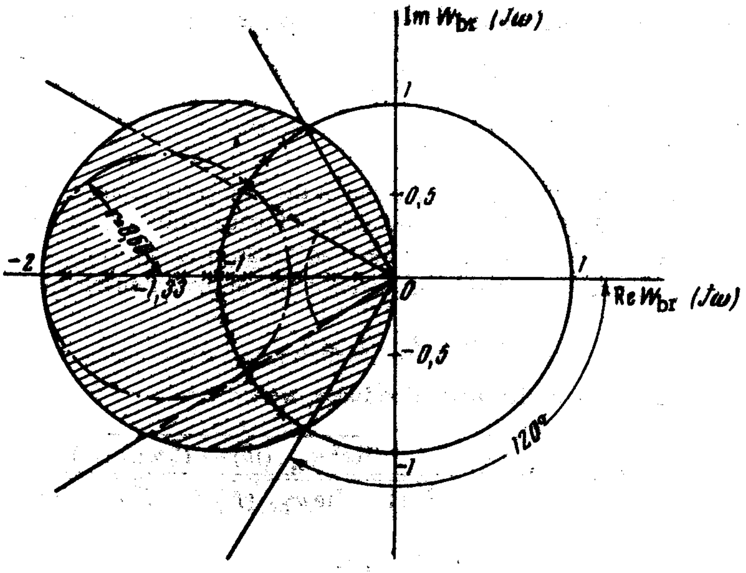

Fig. 2

We investigate the dependence of ε on μ_0^2. This dependence has a nonexplicit character, since the quantity $r'(Q + A)r$ depends on μ_0^2. However, if the plant (1.1) is asymptotically stable, then it is not difficult to show that for every μ_0^2 the inequality $r'(Q + A)r \leq a$ is satisfied, where a is a positive number that is independent of μ_0^2.

A consequence of the boundedness of $r'(Q + A)r$ is the relation

$$\varepsilon \to 0 \quad \text{as} \quad \mu_0^2 \to \infty. \tag{2.10}$$

We describe the frequency properties of the optimal discrete systems with a single controller.

Theorem 2.1. If system (1.1), (1.2) with scalar control and asymptotically stable plant (1.1) is optimal in the sense of the functional (2.1), then for its frequency performance indices the following estimates hold:

$$\text{a) } \varphi_M \geqslant 60^\circ - \varepsilon_1, \quad \text{b) } L \geqslant 2 - \varepsilon_2, \quad \text{c) } M \leqslant 2 + \varepsilon_3, \tag{2.11}$$

in which the positive numbers ε_1^0, ε_2, and ε_3 are such that for any numbers ε_1^{0*}, ε_2^*, and ε_3^* given beforehand, there always exists a sufficiently large number μ_0^2, such that

$$\varepsilon_i \leqslant \varepsilon_i^*. \tag{2.12}$$

Proof. We represent the identity (2.8) in the form

$$1 + 2\,\mathrm{Re}\, w_{br}(jv) + \mathrm{Re}^2 w_{br}(jv) + I_m^2 w_{br}(jv) = \tag{2.13}$$

$$= 1 - \varepsilon + (1 - \varepsilon)\mu_0^{-2} \sum_{i=1}^{m} h_i(-jv) h_i(jv).$$

Taking into account the relation

$$\varepsilon \leqslant 1, \qquad \sum_{i=0}^{m} h_i(-jv) h_i(jv) \geqslant 0, \tag{2.14}$$

we obtain

$$[\mathrm{Re}\, w_{br}(jv) + 1]^2 + I_m^2 w_{br}(jv) \geqslant 1 - \varepsilon. \tag{2.15}$$

The equality $[\mathrm{Re}\, w_{br}(jv) + 1]^2 + I_m^2 w_{br}(jv) = 1 - \varepsilon$ corresponds to a circle of radius $\sqrt{1-\varepsilon}$ with center at the point

$$\mathrm{Re}\, w_{br}(jv) = -1, \qquad I_m w_{br}(jv) = 0. \tag{2.16}$$

This circle is the boundary of the disk forbidden for the amplitude–phase characteristics of the optimal system (1.1), (1.2).

We first consider the limiting case

$$\varepsilon = 0. \tag{2.17}$$

The forbidden disk that corresponds to this case is noted in the hodograph plane of the amplitude–phase characteristics, shown by hatching in Fig. 2. On this figure the circle of unit radius with center at the coordinate origin is shown. It is not difficult to see that the arc $[-120° + 2\psi\pi, -240° - 2\psi\pi]$ ($\psi = 0, 1, \ldots$) of the latter circle, and also the segment $[-2,0]$ of the real axis, occur inside the hatched disk of the forbidden zone. The latter fact indicates that

$$\varphi_M \geqslant 60°, \quad L \geqslant 2. \tag{2.18}$$

In order to determine the boundary of the oscillation index, we write

$$M = \max M(v), \quad M(v) = \frac{\sqrt{\operatorname{Re}^2 w_{br}(jv) + I_m^2 w_{br}(jv)}}{\sqrt{[1 + \operatorname{Re} w_{br}(jv)]^2 + I_m^2 w_{br}(jv)}}. \tag{2.19}$$

$$0 \leqslant v \leqslant \infty$$

Squaring the last equation and eliminating the denominator [2], we obtain after algebraic transformations

$$\left[\operatorname{Re} w_{br}(jv) + \frac{M^2(v)}{M^2(v)-1}\right]^2 + I_m^2 w_{br}(jv) = \left[\frac{M(v)}{M^2(v)-1}\right]^2. \tag{2.20}$$

Let $M(v) = \text{const} = \bar{M}$; then the last equation describes a circle of radius $r = \bar{M}/\bar{M}^2-1$ with center at the point $(-a, j0)$, where $a = \bar{M}/\bar{M}^2-1$, being the geometrical position of the points forbidden for intersection of the amplitude–phase characteristics of systems with oscillation index $M \leq M$. In Fig. 2, by a dot-dash line we plot the circle corresponding to $\bar{M} = 2$ ($r = 0.66$, $a = 1.33$), which occurs inside the hatched disk, and, hence

$$M \leqslant 2. \tag{2.21}$$

The analytical proof of inequalities (2.18) and (2.21) is presented in [6].

In the fundamental case ($\varepsilon \neq 0$) the radius of the disk forbidden for the amplitude–phase characteristics of the optimal system decreases with increasing ε, and therefore the boundaries of the frequency performance indices fall outside the limits indicated by the inequalities (2.18) and (2.21). In this case the boundaries of the frequency performance indices can be described by expressions (2.11). Taking account of (2.10), we conclude that with increasing μ_0^2 the boundaries (2.11) converge to the boundaries (2.18) and (2.21), which also proves Theorem 2.1.

We now generalize this theorem to the case of systems with m controllers ($m \geq 1$).

We represent system (1.1), (1.2) in the form of the m following equivalent systems:

$$x[(k+1)T] = [\Phi + \overset{i\,i}{R C'}]\, x(kT) + R_{[i]} u_i(kT) \quad (i = 1, \ldots, m), \tag{2.22}$$

$$u_i(kT) = C'_{[i]} x(kT) \quad (i = 1, \ldots, m), \tag{2.23}$$

and we consider the ν-th system. (The symbol [i] indicates the i-th column of the matrix.)

Taking into account the optimality of system (1.1), (1.2), we write, on the basis of (2.3),

$$\overset{\nu}{C'} = -\overline{\{R\,[E\mu_0^2 + R'(Q+A)R]^{-1}\}}(Q+A)\Phi, \tag{2.24}$$

$$C'_{[\nu]} = -\{R\,[E\mu_0^2 + R'(Q+A)R]^{-1}\}'_{[\nu]}(Q+A)\Phi.$$

The transfer function of system (1.1), (1.2), broken at the ν-th input of the plant, has, in conformity with (2.22) and (2.23), the form

$$w_\nu(z) = \{R\,[E\mu_0^2 + R'(Q+A)R]^{-1}\}_{[\nu]}(Q+A)\Phi \; \times \tag{2.25}$$
$$\times\, [Ez - \Phi + \overset{\nu}{R}\,\overline{\{R\,[E\mu_0^2 + R'(Q+A)R]^{-1}\}}'(Q+A)\Phi]^{-1} R_{[\nu]}.$$

We consider the following auxiliary problem.

Problem 2.2. Let there be a control plant, described by the equation

$$x[(k+1)T] = \Phi^* x(kT) + R^* u^*(kT), \tag{2.26}$$

where

$$\Phi^* = \Phi + \check{R}\check{C}' = \Phi - \check{R}\{[E\mu_0^2 + \check{R}'(Q+A)\check{R}]^{-1}\}'(Q+A)\Phi, \quad R^* = R_{[\nu]}.$$

It is required to determine the controller equation

$$u^*(kT) = c^{*\prime} x(kT) \tag{2.27}$$

such that for asymptotically stable motions of system (2.26), (2.27), excited by arbitrary initial deviations, the following functional is minimized:

$$I = \sum_{k=0}^{\infty} x'(kT)\, Q^* x(kT) + \mu_0^2 u^{*2}(kT), \tag{2.28}$$

where

$$Q^* = Q + \mu_0^2 \check{C}\check{C}' = + \mu_0^2 \Phi'(Q+A) \times$$
$$\times \{\check{R}[E\mu_0^2 + \check{R}'(Q+A)\check{R}]^{-1}\}\{\check{R}[E\mu_0^2 + \check{R}'(Q+A)\check{R}]\}'(Q+A)\Phi.$$

Let this problem be solved. It turns out that the following equation holds:

$$(Q^* + A^*) = (Q + A), \quad c^* = c_{[\nu]}. \tag{2.29}$$

The proof of these equations (a similar proof of the corresponding relations for continuous systems is presented in [7]), which, in principle, is complicated, appears in a relatively large number of works and therefore will not be presented here.

The transfer function

$$w^*_{\text{br}}(z) = -c^{*\prime}[Ez - \Phi^*]^{-1} R^* \tag{2.30}$$

of the optimal system (2.26), (2.27) in the broken state satisfies the identity

$$[1 + w^*_{\text{br}}(z^{-1})][1 + w^*_{\text{br}}(z)] = \tag{2.31}$$

$$= (1 - \varepsilon^*)\left[1 + \mu_0^{-2} \sum_{i=1}^{n} h_i^*(z^{-1}) h_i^*(z)\right],$$

where the $h_i^*(z)$ ($i = 1, \ldots, n$) are components of the n-dimensional vector

$$H^* = H^*(E - \Phi^*)^{-1} R^*, \quad H^{*\prime}H^* = Q^*,$$

$$\varepsilon^* = \frac{R^*(Q^* + A^*)R^*}{\mu_0^2 + R^*(Q^* + A^*)R^*}.$$

Using Eq. (2.29), we conclude on the basis of (2.25) and (2.30) that

$$w^*_{\text{br}}(z) = w_\nu(z). \tag{2.32}$$

Furthermore,

$$\varepsilon^* = \varepsilon_\nu = \frac{R_{[\nu]}(Q + A)R_{[\nu]}}{\mu_0^2 + R_{[\nu]}(Q + A)R_{[\nu]}}.$$

Taking account of (2.32) we write the identity (2.31) in the form

$$[1 + w_\nu(z^{-1})][1 + w_\nu(z)] = \tag{2.33}$$

$$= (1 - \varepsilon_\nu)\left[1 + \mu_0^{-2}\sum_{i=1}^{n} h_i^\nu(z^{-1})h_i^\nu(z)\right].$$

It is not difficult to verify that

$$\varepsilon_\nu \to 0 \quad \text{as} \quad \mu_0^{-2} \to 0. \tag{2.34}$$

Repeating for (2.33) the proof of Theorem 2.1, we obtain estimates of the form (2.11).

What has been discussed above holds for all ν from 1 to m and thus the following theorem is proved.

Theorem 2.2. If the control plant (1.1) of the system (1.1), (1.2) which is optimal in the sense of the functional (2.1), is asymptotically stable, then for frequency performance indices of this system the following estimates hold:

$$\text{a) } \varphi_{Mi} \geqslant 60° - \varepsilon_{1i}^{\circ}, \quad \text{b) } L_i \geqslant 2 - \varepsilon_{2i}, \quad \text{c) } M_i \leqslant 2 + \varepsilon_{3i} \quad (i = 1, \ldots, m), \tag{2.35}$$

in which the numbers ε_{1i}, ε_{2i}, and ε_{3i} ($i = 1, \ldots, m$) are positive and vanish for sufficiently large values of μ_0^2.

3. Matrix of Transfer Constants and Frequency Performance Indices of Optimal Systems

Taking account of the boundedness of the matrix A, we can write the identity (2.6) for sufficiently large μ_0^2 in the form

$$[E + W_{br}(z^{-1})]'[E + W_{br}(z)] \approx E + \mu_0 H'(z^{-1})H(z). \tag{3.1}$$

Assuming z = 1, we obtain the relation

$$[E + W_{br}(1)]'[E + W_{br}(1)] \approx E + \mu_0 H'(1)H(1). \tag{3.2}$$

For m = 1 this relation has the form

$$(1 + k_p)^2 = 1 + \mu_0^{-2}\sum_{i=1}^{n} h_i^2(1). \tag{3.3}$$

It follows directly from (3.2) or (3.3) that a sufficiently large value of the parameter μ_0^2 (necessary for guaranteeing, in conformity with Theorems 2.1 and 2.2, sufficiently good performance indices characterizing the dynamics of the transition process) corresponds to small values of the matrix elements of the transfer constants. In connection with this contradiction between the required static accuracy of system (1.1), (1.2) and the performance indices characterizing the dynamics, we consider the following problem.

Problem 3.1. Let there be a system of equations

$$x[(k+1)T] = \Phi x(kT) + Ru(kT), \tag{3.4}$$

$$\tilde{x}[(k+1)T] = (1 - t_{1g}^{-1})\tilde{x}(kT) + t_{1g}Dx(kT), \tag{3.5}$$

where $\tilde{x}(kT)$ is an r-dimensional vector ($r \leq n$), $t_{1g} > 1$, D is the matrix of the numbers of dimension $r \times n$, satisfying the condition

$$\text{rank}\|D'(DP' \ldots (DP^{n-1})'\| = n. \tag{3.6}$$

It is required to determine the control

$$u(kT) = \check{C}'x(kT) + C'\tilde{x}(kT) \tag{3.7}$$

such that on solutions of the system (3.4), (3.5), (3.7) the following functional is minimized:

$$I = \sum_{k=0}^{\infty} \tilde{x}'(kT)Q_1\tilde{x}(kT) + u'(kT)u(kT). \tag{3.8}$$

This problem and its solution differ from Problem 2.1 only in the dimensions of the vector x. Actually, introducing the notation

$$\bar{x} = \begin{matrix} n\{ \\ r\{ \end{matrix}\left\| \begin{matrix} x \\ z \end{matrix} \right\|, \quad \bar{\Phi} = \left\| \begin{matrix} \Phi & 0 \\ t_{1g}^{-1}DE\,(1-t_{1g}^{-1}) \end{matrix} \right\|, \tag{3.9}$$

$$\bar{R} = \begin{matrix} n\{ \\ r\{ \end{matrix}\left\| \begin{matrix} R \\ 0 \end{matrix} \right\|, \quad \bar{Q} = \begin{matrix} n\{ \\ r\{ \end{matrix}\left\| \begin{matrix} 0 & 0 \\ 0 & Q_1 \end{matrix} \right\|,$$

we solve Problem 2.1, in which Eq. (1.1) and functional (2.1) have the form

$$\bar{x}[(k+1)T] = \bar{\Phi}\bar{x}(kT) + \bar{R}u(kT), \tag{3.10}$$

$$I = \sum_{k=0}^{\infty} \bar{x}'(kT)\bar{Q}\bar{x}(kT) + u'(kT)u(kT). \tag{3.11}$$

As a result of the solution we obtain

$$u(kT) = \bar{C}'\bar{x}(kT), \tag{3.12}$$

where

$$\bar{C}' = \|\check{C}'C'\|. \tag{3.13}$$

The matrix of the transfer constants of the optimal system (3.10), (3.12) has the form

$$W_{\text{br}}(z) = -\bar{C}'(Ez-\bar{\Phi})^{-1}\bar{R} \tag{3.14}$$

and satisfies the identity

$$\begin{aligned} \bar{N} + \bar{N}W_{\text{br}}(z) + W'_{\text{br}}(z^{-1})\bar{N} + W_{\text{br}}(z^{-1})\bar{N}W_{\text{br}}(z) \\ = E + \bar{H}'(z^{-1})\bar{H}(z), \end{aligned} \tag{3.15}$$

where

$$\bar{N} = \bar{R}'(\bar{Q}+\bar{A})\bar{R} + E. \tag{3.16}$$

We express the matrices that appear in (3.15) and (3.16) in terms of blocks of the matrices Φ, $\bar{R}$, and $\bar{Q}$. Using (3.9), we obtain

$$(Ez-\bar{\Phi})^{-1} = \left\| \begin{matrix} (Ez-\Phi)^{-1} & 0 \\ t_{1g}^{-1}D(Ez-\Phi)^{-1}(z-1+t_{1g})^{-1} & E(z-1+t_{1g}^{-1})^{-1} \end{matrix} \right\|$$

$$(Ez-\bar{\Phi})^{-1}\bar{R} = \left\| \begin{matrix} (Ez-\Phi)^{-1}R \\ t_{1g}^{-1}(z-1+t_{1g}^{-1})^{-1}D(Ez-\Phi)^{-1}R \end{matrix} \right\|,$$

$$\begin{aligned} \bar{H}'(z^{-1})\bar{H}(z) = R'(Ez-\Phi)^{-1}D'Q_1D(Ez-\Phi)^{-1}\times \\ \times R\{t_{1g}^{-2}(z-1+t_{1g}^{-1})^{-1}(z^{-1}-1+t_{1g}^{-1})^{-1}\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} W_{\text{br}}(z) = -\check{C}'(Ez-\Phi)^{-1}R - \\ - C't_{1g}^{-1}(z-1+t_{1g}^{-1})^{-1}D(Ez-\Phi)^{-1}R. \end{aligned} \tag{3.18}$$

Representing the matrix A in the form

$$A = \left\| \begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix} \right\|, \quad A'_{12} = A_{21},$$

we obtain

$$\bar{N} = R'A_{11}R + E. \tag{3.19}$$

The following relation holds:

$$A_{11} \to [0] \quad \text{as} \quad t_{1g}^{-1} \to 0, \tag{3.20}$$

using this we can write (3.15) in the form

$$[1+\overline{W}_{br}(z^{-1})]'[1+\overline{W}_{br}(z)] \approx E + \overline{H}'(z^{-1})\overline{H}(z). \tag{3.21}$$

It is not difficult to verify, repeating the fundamental propositions of the proofs of Theorems 2.1 and 2.2, that the frequency performance indices of system (3.10), (3.12) satisfy the inequalitites (2.35), in which the numbers ε_{ji} ($j = 1, 2, 3; i = 1, \ldots, m$) go to zero with increasing t_{1g}.

The rigorous proof of relation (3.20) is rather cumbersome and therefore will not be given here.

Remark. We present an intuitive explanation, confirming the validity of the relation (3.20).

The "artificial" plant (3.10) differs from the "real" plant (3.4) by the presence of r inertial sections (3.5). With increasing inertia of these sections (with increasing parameter t_{1g}) the effect of the vector x (and, in particular, its initial values) decreases for processes based on the vector $\tilde{x}$, and the matrix A_{11} characterizes the weight of the vector $x(0)$ in the minimum value of the functional (3.8), since $I_{opt} = x'(0)A_{11}x(0) + x'(0)A_{12}\tilde{x}(0) + \tilde{x}'(0)A_{21}x(0) + \tilde{x}'(0)A_{22}\tilde{x}(0)$.

We now eliminate the vector $\tilde{x}(kT)$ from Eq. (3.12), using (3.5). Then

$$u(z) = -(\check{C}' + \tilde{C}'t_{1g}^{-1}(z-1+t_{1g}^{-1})D]x(z). \tag{3.22}$$

We investigate the properties of the plant (3.4), closed by the controller (3.22).

The transfer matrix $\tilde{W}_{br}(z)$ of system (3.4), (3.22) has the form

$$\tilde{W}_{br}(z) = -[\check{C}' + \tilde{C}t_{1g}^{-1}(z-1+t_{1g}^{-1})D](Ez-\Phi)^{-1}R. \tag{3.23}$$

Comparing this matrix with (3.18), we conclude that

$$\tilde{W}_{br}(z) \equiv \overline{W}_{br}(z). \tag{3.24}$$

An almost obvious consequence of this identity is the relation

$$\tilde{w}_i(z) \equiv \overline{w}_i(z) \quad (i = 1, \ldots, m), \tag{3.25}$$

where $\overline{w}_i(z)$ and $\tilde{w}_i(z)$ are the transfer functions of the systems (3.10), (3.12) and (3.4), (3.22), respectively, broken with respect to the i-th input of the plant. The identity (3.25) indicates agreement of the frequency performance indices of these systems.

Using (3.17) and (3.24), based on (3.21) we write

$$[E+\tilde{W}_{br}(z^{-1})]'[E+\tilde{W}_{br}(z)] \tag{3.26}$$

$$= E + \tilde{H}'(z^{-1})\tilde{H}(z)\{t_{1g}^{-1}(z^{-1}-1+t_{1g}^{-1})^{-1}(z-1+t_{1g}^{-1})^{-1}\},$$

where $\tilde{H}(z) = H_1D(Ez-\Phi)^{-1}R$, $H_1'H_1 = Q_1$.

For z = 1 we obtain

$$[E+\tilde{W}_{br}(1)]'[E+\tilde{W}_{br}(1)] = E + \tilde{H}'(1)\tilde{H}(1). \tag{3.27}$$

Thus, choosing in Problem 3.1 sufficiently large values of the parameter t_{1g} and the elements of matrix Q_1, we obtain controllers that guarantee given performance indices for the system (3.4), (3.22).

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