DETERMINATE SYSTEMS

PROPERTIES OF ANALYTICALLY CONSTRUCTED LINEAR SYSTEMS

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UDC 62-50

Frequency and transient properties are investigated for systems whose controls are obtained via known analytical construction algorithms. A set of properties is established for such systems, where the set does not depend on the choice of the coefficients in the nonnegative optimization functional.

1. The Problem

The wide-ranging possibilities in the analytical approach to control synthesis developed in the last decades have attracted the attention of engineers designing controls for actual systems.

In many cases, and in particular for complex multidimensional systems, this approach is eminently feasible. But its application is often difficult, due to the well-known problem of the selection of the coefficients in the optimization functional used in the mathematical description of the nature of the control. On the other hand, practical control synthesis usually is achieved through quality frequency characteristics, and also through quality characteristics of the curve of the transient process [1, 2].

Thus it is important to establish the boundaries of the above quality characteristics for analytically constructed systems, where these boundaries do not depend on the choice made for the coefficient in the optimization functional. This article investigates such properties.

Suppose we are given a control system whose perturbed motion is described by the following equations, in a first approximation:

$$\dot{x}=Px+Bu, \tag{1.1}$$

$$u=C'x, \tag{1.2}$$

where x is the n-dimensional phase coordinate vector for the object, u is the output variable of the regulator (the control), P, B are given real matrices of dimensions $n \times n$ and $n \times 1$, respectively, and they characterize the controllable object completely, C is an n-dimensional column vector, and the prime denotes the transpose operation.

The problem of analytical construction of controls was formulated and solved by Letov in 1960 [3]; it consists of the determination of the vector C such that in asymptotically stable motions of the system (1.1), (1.2) perturbed by arbitrary initial deviations the following functional is minimized:

$$I=\int_0^\infty (x'Qx+u^2)\,dt \tag{1.3}$$

for the given nonnegative definite matrix Q.

The matrix Q can be written in the form $Q=H'H$, where H is a matrix of dimensions $r \times n$ (r is the rank of the matrix Q). It is assumed further that the matrix H satisfies the condition

Saratov. Translated from Avtomatika i Telemekhanika, No. 10, pp. 5-11, October, 1975. Original article submitted November 22, 1974.

$$\det\|H'HP\ldots HP^{n-1}\|\neq 0. \tag{1.4}$$

The procedure for solving the analytical construction problem for controls, called below Procedure 1, consists, as is well known, of two operations: first solve the equations

$$A^{(1)}P+P'A^{(1)}-A^{(1)}BB'A^{(1)}+Q=0 \tag{1.5}$$

and then calculate the desired vector C from the formulas

$$C=C^{(1)}=-A^{(1)}B, \tag{1.6}$$

where $A^{(1)}$ is a positive definite symmetric $n\times n$ matrix. In 1967 Krasovskii [4] proposed a computational improvement to Procedure 1.

In this case, the object is assumed asymptotically stable, and the functional (1.3) contains the additional term:

$$\frac{1}{4}\int_0^\infty\left[\sum_{i=1}^n\frac{\partial V_2}{\partial x_i}b_i\right]^2dt,$$

in which the positive definite quadratic form $V_2 = x'A^{(2)}x$ is the solution of the linear partial differential equation

$$\sum_{i=1}^n\frac{\partial V_2}{\partial x_i}\left[\sum_{j=1}^n p_{ij}x_j\right]=-\sum_{i,j=1}^n q_{ij}x_ix_j.$$

The known procedure for solving the analytically constructed control problem in this case still consists of two operations: first solve the equations

$$A^{(2)}P+P'A^{(2)}+Q=0 \tag{1.7}$$

and then calculate the vector

$$C=C^{(2)}=-A^{(2)}B, \tag{1.8}$$

where $A^{(2)}$ is a positive definite symmetrix $n\times n$ matrix.

This sequence for operations will be called Procedure 2.

We note that equations (1.5) and (1.7) coincide formally, if in place of the matrix Q in (1.5) we put the matrix $Q-A^{(1)}BB'A^{(1)}$.

Here and in what follows below the upper indices (1) and (2) attached to matrices, vectors, and scalars denote the number of the Procedure from which the matrices and scalars arise.

Procedures 1 and 2 use the nonnegative definite matrix Q.

The aim of the following sections is an investigation of the frequency and transient process properties of systems of the form (1.1), (1.2), whose regulators (1.2) are obtained from Procedures 1 and 2 where the properties are not to depend on the choice made for the elements of the nonnegative matrix Q.

2. Frequency Properties of Analytically Constructed Systems

First we quote the definition of the transfer function of the system (1.1), (1.2) in its open state, and then we establish the relations between the transfer functions of this system with matrix Q for the cases where the control (1.2) is obtained through Procedures 1 and 2. These relations are basic for the investigation of the frequency properties of analytically constructed systems.

As transfer function for the system (1.1), (1.2) in the open state we may take the function relating the Laplace transform of the output variable u of the control and the disturbance r fed into the input of the object instead of this variable. The system (1.1), (1.2) is described in this case by the equations $sx = Px + Br$, $u = C'x$ (s is the symbol for the Laplace transform under zero initial conditions).

It follows from these equations that $x = (Fs-P)^{-1}Br$, and $u = C'(Es-P)^{-1}Br$ and, so that

$$w_{op}(s) = -C'(Es-P)^{-1}B. \tag{2.1}$$

Suppose the vector C in (1.2) has been obtained through Procedure 1. Then in this case the relation between the transfer function of the system (1.1), (1.2) and the matrix Q is well known [5]. This relation is called (when $s = j\omega$) the optimality condition infrequency form, and is

$$[1+w_{op}^{(1)}(s)][1+w_{op}^{(1)}(-s)] = 1+H'(-s)H(s), \tag{2.2}$$

where

$$H(s) = H(Es-P)^{-1}B, \quad w_{op}^{(1)}(s) = -C'(Es-P)^{-1}B.$$

Let us now introduce the analogous relation for systems whose control (1.2) has been obtained from Procedure 2. Let us add and subtract from the left side of (1.7) the term $sA^{(2)}$, and then let us premultiply this relation by $B'(-Es-P)^{-1}$ and postmultiply it by $B(Es-P)^{-1}$. Then $B'(-Es-P)^{-1'}[(-Es-P)' - A^{(2)} + A^{(2)}(Es-P)](Es-P)^{-1}B = B'(-Es-P)^{-1}H'H(Es-P)^{-1}B$.

Using (1.8) and (2.1) we get the identity

$$w_{op}^{(2)}(s) + w_{op}^{(2)}(-s) = H'(-s)H(s). \tag{2.3}$$

With $s = j\omega$ we write

$$w_{op}^{(2)}(j\omega) + w_{op}^{(2)}(-j\omega) = H'(-j\omega)H(j\omega). \tag{2.4}$$

This identity for all real ω can be called the optimality condition in frequency form for the system (1.1), (1.2) in the sense of the functional

$$I = \int_0^\infty \left\{ x'Qx + \frac{1}{4}\left[\sum_{i=1}^{n} \frac{\partial V_2}{\partial x_i} b_i\right]^2 + u^2 \right\} dt.$$

Let us now investigate the frequency properties of analytically constructed systems through the identities (2.2) and (2.4).

Theorem 1. The hodograph of the amplitude-phase frequency characteristic of the system (1.1), (1.2) whose control is obtained through Procedure 1, for an arbitrary nonnegative definite matrix Q, does not intersect the zone bounded by the circumference of the circle of unit radius centered at the point (−1, j 0).

This means that the frequency qualitative indices of this system (phase range φ_r, modulus range L, vibration index M) satisfy the inequalities

$$\varphi_r \geqslant 60°, \quad L \geqslant 2, \quad M \leqslant 2. \tag{2.5}$$

Proof. Write (2.2) with $s = j\omega$ in the form:

$$[1 + \mathrm{Re}\, w_{op}^{(1)}(j\omega)]^2 + \mathrm{Im}^2\, w_{op}^{(1)}(j\omega) = 1 + \sum_{i=1}^{n} h_i(-j\omega)h_i(j\omega), \tag{2.6}$$

where $h_i(j\omega)$ $(i = 1, \ldots, n)$ are the components of the vector $H(j\omega) = H(Ej\omega - P)^{-1}B$. It is obvious that

$$\sum_{i=1}^{n} h_i(-j\omega)h_i(j\omega) \geqslant 0, \tag{2.7}$$

and therefore

$$[1 + \mathrm{Re}\, w_{op}^{(1)}(j\omega)]^2 + \mathrm{Im}^2\, w_{op}^{(1)}(j\omega) \geqslant 1. \tag{2.8}$$

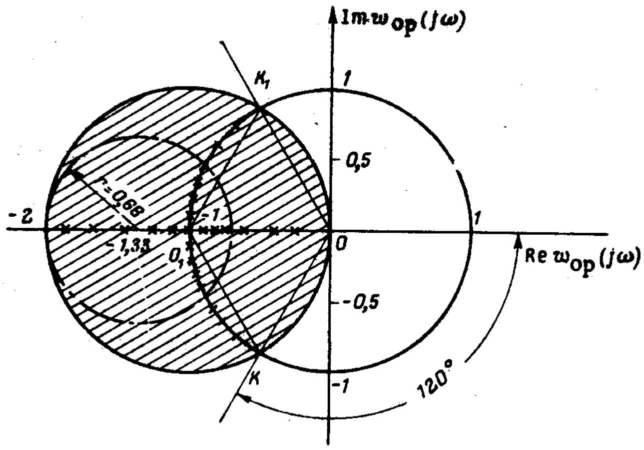

Fig. 1

To the equation $[\mathrm{Re}\, w^{(1)}_{op}(j\omega)+1]^2+\mathrm{Im}^2 w^{(1)}_{op}(j\omega)=1$ there corresponds the circumference of the unit circle centered at the point $\mathrm{Re}\, w^{(1)}_{op}(j\omega)=-1$, $\mathrm{Im}\, w^{(1)}_{op}(j\omega)=0$, in the $\mathrm{Re}\, w_{op}(j\omega)$, $\mathrm{Im}\, w_{op}(j\omega)$ plane, and therefore the first part of the theorem is proved.

This circle of unit radius which lies in the forbidden zone for the amplitude-phase frequency characteristics of the systems (1.1), (1.2) obtained through Procedure 1 is shaded in Fig. 1.

The geometrical figures of Fig. 1 make verification of (2.5) easy.

In fact, the forbidden zone includes the following: the arc $K_1 0_1 K$ (characterizing the stable phase range) of 120°, since the triangles $0_1 0 K$ and $0_1 0 K_1$ are equilateral; the circumference corresponding to the vibration index $M = 2$; and the interval $[-2, 0]$ on the real axis. We note that this last implies that the modulus range for systems with amplitude-phase frequency characteristics of the second kind is not less than two, and for systems of the first kind is infinitely large.

The analytical basis for the properties (2.5) is given in [6].

Theorem 2. The hodograph of the amplitude-phase frequency characteristics of the system (1.1), (1.2) whose control (1.2) is found through Procedure 2 does not leave the first and fourth quadrants of the plane

$$\mathrm{Re}\, w_{op}(j\omega),\ \mathrm{Im}\, w_{op}(j\omega)$$

for any arbitrary nonnegative matrix Q.

This means that the qualitative indices of the system (1.1), (1.2) satisfy the relations:

$$\varphi_r \geqslant 90°,\ L \to \infty,\ M \leqslant 1. \qquad (2.9)$$

Proof. Write (2.4) in the form:

$$2\,\mathrm{Re}\, w^{(2)}_{op}(j\omega)=\sum_{i=1}^{n} h_i(-j\omega)\,h_i(j\omega).$$

Using (2.7) we get the inequality:

$$2\,\mathrm{Re}\, w^{(2)}_{op}(j\omega) \geqslant 0, \qquad (2.10)$$

from which follows the assertion of the first part of the theorem.

To prove the first of the relations in (2.9), we write (2.10) in the form $2A^{(2)}(\omega)\cos\varphi^{(2)}(\omega) \geq 0$. Now using $A^{(2)}(\omega_{av}) = 1$ we get $\cos\varphi^{(2)}(\omega_{av}) \geq 0$.

This means that the phase range of the system (1.1), (1.2) is not less than 90°.

The second inequality in (2.9) is obvious.

The last of these inequalities is proved as follows: we recall from [2] that

$$M = \max_{0 \le \omega \le \infty} \left| \frac{w_{op}(j\omega)}{1+w_{op}(j\omega)} \right|.$$

Write this relation in the form $M^2 = \max_{0 \le \omega \le \infty} M^2(\omega)$, where

$$M^2(\omega) = \frac{w_{op}(j\omega)\, w_{op}(-j\omega)}{[1+w_{op}(j\omega)][1+w_{op}(-j\omega)]}.$$

In the case under consideration $w_{op}(j\omega)$ satisfies (2.4) and therefore

$$M^2(\omega) = \frac{A^{(2)^2}(\omega)}{1+w_{op}^{(2)}(j\omega)+w_{op}^{(2)}(-j\omega)+A^{(2)^2}(\omega)} = \frac{A^{(2)^2}(\omega)}{1+\sum_{i=1}^{n} h_i(-j\omega)\, h_i(j\omega)+A^{(2)^2}(\omega)}$$

From (2.7) we conclude that $M \le 1$.

We note that the proof of the properties given in these theorems comes from the inequalities (2.8) and (2.10) which do not contain the matrix Q; therefore the given properties hold for all nonnegative definite matrices Q.

Let us now investigate the properties of transfer functions in the optimal system (1.1), (1.2) for the unit step disturbance r(t). This disturbance excites various transient processes depending on where r(t) is applied and which variable is investigated.

By a transient process in the system (1.1), (1.2) we mean a process in the variable u(t), a process perturbed by a unit action applied to the object input. Moreover, for the particular case of the object (1.1) with scalar output $y = x$, and $x_2 = \dot{y}, \ldots, x_n = y^{(n-1)}$, we shall take a transient process in the system (1.1), (1.2) to be a process in the variable y(t) perturbed by r(t) applied to the input of the control.

The transient processes being investigated are determined by the nature of the real frequency characteristic of the closed system $R(\omega)$.

Corollary 1. If the control (1.2) is obtained through Procedure 1, then

$$2 \ge R^{(1)}(\omega) \ge 0. \tag{2.10'}$$

Usually $R(\omega)$ is a continuous function with a single extremum, a maximum, and the amplification coefficient if the open system is $K_p = w_{op}^{(1)}(0) \gg 1$. In this case (2.10') implies that the overshoot in the system (1.1), (1.2) does not exceed 136%.

Proof. If we put

$$w_c(j\omega) = \frac{w_{op}(j\omega)[1+w_{op}(-j\omega)]}{[1+w_{op}(j\omega)][1+w_{op}(-j\omega)]},$$

we get

$$R(\omega) = \frac{\operatorname{Re} w_{op}(j\omega)+A^2(\omega)}{|1+w_{op}(j\omega)|^2}. \tag{2.11}$$

Rewrite (2.2) in the form $\operatorname{Re} w_{op}^{(1)}(j\omega) = \frac{1}{2}\left[\sum_{i=1} h_i(-j\omega)\, h_i(j\omega) - A^{(1)^2}(\omega)\right]$ and eliminate $\operatorname{Re} w_{op}(j\omega)$ from (2.11) we get $R^{(1)}(\omega) \ge 0$.

From the last of the inequalities in (2.5) we conclude that $R^{(1)}(\omega) \leq 2$.

To estimate the value of the overshoot $\sigma\%$, we use the well known [1] formula $\sigma \leq \{[1.18R_m - R(0)]/R(0)\}$. If we take account of the fact that $R(0) = w_c^{(1)}(0) = w_{op}^{1}(0)[1 + w_{op}^{(1)}(0)]$, we get $\sigma \leq 2.36/K_p + 1.36$. Using $K_p = w_{op}(0)$ >> 1, we get $\sigma = 136\%$.

Corollary 2. If the control (1.2) is found through Procedure 2, then

$$1 \geqslant R^{(2)}(\omega) \geqslant 0. \tag{2.12}$$

If there is no extremum in the characteristic $R^{(2)}(\omega)$ then the overshoot $\sigma \leq 18\%$.

In fact, from (2.11) and (2.10) we may conclude that $R^{(2)}(\omega) \geq 0$.

From the last inequality of (2.9) it follows that $R^{(2)}(\omega) \leq 1$, and the formula for σ implies that $\sigma \leq 18\%$.

3. Properties of Multidimensional Systems

Suppose that the matrices B and C in the system (1.1), (1.2) are of dimension $n \times m$. For convenience we shall say this system is one-dimensional for m = 1, and multidimensional for m > 1. The synthesis procedures 1 and 2 also work for this more general case. We write the multidimensional system (1.1), (1.2) as a one-dimensional one of the following form:

$$\dot{x} = [P + \overset{\nu}{B}\overset{\nu}{C}']x + B_{[\nu]}u_\nu, \tag{3.1}$$

$$u_\nu = C'^{[\nu]}x, \tag{3.2}$$

where $B_{[\nu]}$ is the ν-th column of the matrix B, $C'[\nu]$ is the ν-th row of the matrix C', $\overset{\nu}{B}$ is a matrix of dimensions $n \times (m-1)$ derived from the matrix B by deleting the ν-th column, and $\overset{\nu}{C}'$ is the matrix of dimensions $(m-1) \times n$ derived from C' by deleting the ν-th row.

The object (3.1) is this object closed by all controls except the ν-th. The transfer function of the system (3.1), (3.2), in accordance with the expression (2.1), has the form:

$$w_\nu(s) = -C'^{[\nu]}(Es - P^*)^{-1}B_{[\nu]}, \tag{3.3}$$

where $P^* = P + \overset{\nu}{B}\overset{\nu}{C}'$. This transfer function is called [7] the transfer function of the system (1.1), (1.2) open at the ν-th input of the object (1.1).

In the relations (3.1) through (3.3) the parameter ν can take values from 1 through m.

Thus the multidimensional system (1.1), (1.2) is characterized by m transfer functions of the form (3.3).

Each function $w_\nu(j\omega)$ has its corresponding $A_\nu(\omega)$ and $\varphi_{z\nu}(\omega)$, and these can be used to obtain $\varphi_{z\nu}$, L_ν, M_ν.

Thus the multidimensional system is characterized by 3m values φ_{zi}, L_i, M_i (i = 1, ..., m).

Suppose the matrix C of the control (1.2) is found from Procedure 2. Close the object (1.1)m by one of the equations obtained and consider the following system:

$$\dot{x} = P^{(2)*} + B_{[\nu]}u_\nu, \tag{3.4}$$

$$u_\nu = C'^{*}x, \tag{3.5}$$

in which $P^{(2)*} = P + \overset{\nu}{B}\overset{\nu}{C}^{(2)'} = P - \overset{\nu}{B}\overset{\nu}{B}'A^{(2)}$, and C'^* is some n-dimensional vector determined by Procedure 2 when $Q \neq Q^*$, $P = P^{(2)*}$.

Lemma. There always exists a nonnegative matrix Q^* such that the equation

$$A^*P^{(2)*} + P^{(2)*'}A^* = -Q^* \tag{3.6}$$

has the solution $A^* = A^{(2)}$, where $A^{(2)}$ is the solution of equation (1.7), and therefore

$$C'^* = -A^*B_{[\nu]} = -A^{(2)}B_{[\nu]} = C^{(2)'[\nu]}. \tag{3.7}$$

Proof. We show that $Q^{*} = Q + 2\overset{\nu}{C}{}^{(2)}\overset{\nu}{C}{}^{(2)\prime}$. In fact, if we substitute in (3.6) the expressions for the matrices $P^{(2)*}$ and Q^{*} and use the fact that $\overset{\nu}{C}{}^{(2)} = -A^{(2)}\overset{\nu}{B}$, we get $A^{*}[P - BB'A^{(2)}] + [P' - A^{(2)}\overset{\nu}{B}\overset{\nu}{B}{}']A^{*} = -Q - 2A^{(2)}\overset{\nu}{B}\overset{\nu}{B}{}'A^{(2)}$.

When $A^{*} = A^{(2)}$, this equation coincides with (1.7), and this proves the assertion of the lemma.

From the Lemma we can conclude that the one-dimensional system (3.4), (3.5) has the frequency properties described in Theorem 2 when $C^{*\prime} = C^{(2)\prime}[\nu]$, and the transient processes in this system satisfy the estimates of Corollary 2. The analogous assertion has been proved in [7] for the case where the matrix C is obtained through Procedure 1.

Thus the qualitative frequency indices of the multidimensional system (1.1), (1.2) whose control (1.2) is found through emitter Procedure 1 or 2 satisfy respectively the inequalities $\varphi_{zi} \geq 60°$, $L_i \geq 2$, and the maximal deviation in the variable $M_i \leq 2$ $(i = 1, \ldots, m)$, $\varphi_{zi} \geq 90°$, $L_i \to \infty$, $M_i \leq 1$ $(i = 1, \ldots, m)$ under unit perturbation r(t) applied to the ν-th input of the object together with the control u_ν does not exceed 2.36 and 1.18 respectively.

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