PROPERTIES OF ANALYTICALLY DESIGNED NONSTATIONARY LINEAR SYSTEMS

A. G. Aleksandrov

UDC 62-501.45

The concept of transfer operator is introduced, as well as the concept of control sensitivity operator of nonstationary systems. The optimality condition is obtained in operator form. It is shown that the sensitivity to parametric disturbances is low, and a relationship is established between controlled variables and disturbances.

1. Introduction

The analytic design of controllers (ADC) [1-3] is an effective method of design of controllers of linear nonstationary systems. But it involves the complicated task of finding the coefficients of the optimization functionals.

One of the methods of extension of the domain of applicability of ADC procedures is to study properties that do not depend on the choice of the coefficients of the optimization functional, and to establish relationships between the parameters of the functional and the accuracy, realizability, etc. Thus for stationary multivariable optimal linear systems we have obtained [4-6] the phase and absolute-value stability margins, as well as a relationship between the structure and the coefficients of the optimization functional on the one hand, and the accuracy, the bandwidth, the realizability, etc., on the other hand.

In this paper this method is extended to nonstationary optimal systems, i.e., we analyze the structural stability and accuracy of such systems.

The study is based on the optimality condition in operator form obtained below which generalizes the optimality conditions of stationary systems in frequency form [7].

The structural stability is estimated with the aid of the concept of control sensitivity operator. This concept is based on the well-known idea [8] of comparing the response of equivalent open-loop and closed-loop systems subjected to parametric disturbances.

2. Concept of Transfer Operator

Let us consider a control system whose disturbed motion is described in the first approximation by the equations

$$\dot{x}=P(t)x+B(t)u,\ x(t_0)=x^{(0)},\ t\in[t_0,t_1], \tag{1}$$

$$u=C'(t)x, \tag{2}$$

where x is an n-dimensional vector of phase variables of the plant, u is an m-dimensional control vector, and P(t), B(t), and C'(t) are assigned matrices (the prime denotes transposition) of continuous and differentiable (as many times as necessary) functions; the dimension of these matrices is n×n, n×m, and m×n respectively; $x^{(0)}$ is an n-dimensional vector of initial conditions. The plant (1) is completely controllable, and the control law (2) is completely observable.

Let us introduce the concept of transfer operator of the system (1)-(2) in the open-loop state. For this purpose we shall replace in (1) the vector u by a vector r(t) of continuous functions, and we shall write for zero initial conditions the solution of Eq. (1):

$$x(t)=\int_{t_0}^{t}X(t,\tau)B(\tau)r(\tau)d\tau, \tag{3}$$

Saratov. Translated from Avtomatika i Telemekhanika, No. 3, pp. 5-12, March, 1980. Original article submitted May 3, 1979.

where $X(t, \tau)$ is a fundamental matrix [9]. In this case the vector of output variables of the controller will be

$$u=C'(t)\int_{t_0}^{t} X(t,\tau)B(\tau)r(\tau)d\tau. \tag{4}$$

The transfer operator W of the system (1)-(2) in the open-loop state will be the operator connecting the vectors u and r taken with opposite sign. In accordance with (4), this operator will be

$$W=-C^*MB, \tag{5}$$

where C and B are finite-dimensional operators generated by the matrices C(t) and B(t) respectively; the symbol * denotes the conjugate operator, i.e., C^* is the operator generated by the matrix $C'(t)$; the operator M is an integral operator that maps the vector $\psi(t) = B(t)r(t)$ into the vector x in accordance with formula (3).

The operator which is conjugate to the transfer operator of the system (1)-(2) in the open-loop state is defined by

$$W^*=-B^*M^*C, \tag{6}$$

where M^* is the conjugate of the operator M.

For specifying conjugate operators, we shall consider below the space $L_2[t_0, t_1]$ [9] of all continuous vector functions defined in the interval $[t_0, t_1]$ and which have a scalar product $(h^{(1)}(t), h^{(2)}(t)) = \int_{t_0}^{t_1} h^{(1)\prime}(t)h^{(2)}(t)\,dt$ and a norm $\|h(t)\| = \left(\int_{t_0}^{t_1} h'(t)h(t)\,dt\right)^{1/2}$.

It is easy to see that

$$M^*\psi=\int_{t}^{t_1} X'(\gamma,t)\psi(\gamma)d\gamma. \tag{7}$$

Now let us introduce the concept of transfer operator of a multivariable system that is open-loop with respect to the ν-th input of the plant. In this connection let us express the equations (1)-(2) in the following [6] equivalent form:

$$\dot{x}=[P(t)+\check{\bar{B}}(t)\check{\bar{C}}'(t)]x+B(t)_{[\nu]}u_\nu, \tag{8}$$

$$u_\nu=C'^{[\nu]}(t)x, \tag{9}$$

where $B(t)_{[\nu]}$ is the ν-th column of the matrix B(t), $C'(t)^{[\nu]}$ is the ν-th row of the matrix $C'(t)$, $\check{\bar{B}}(t)$ is a matrix of dimension $n\times(m-1)$ obtained from the matrix B(t) by deleting the ν-th column, and $\check{\bar{C}}'(t)$ is a matrix of dimension $(m-1)\times n$ obtained from the matrix $C'(t)$ by deleting the ν-th row.

The "plant" (8) corresponds to the plant (1) closed by all the controllers except the ν-th controller.

The transfer operator of system (8)-(9) in the open-loop state is

$$W_\nu=-C^{[\nu]*}M_\nu B_{[\nu]}, \tag{10}$$

where M_ν is an integral operator (similar to the operator M) that corresponds to the matrix

$$P_\nu(t)=P(t)+\check{B}(t)\check{C}'(t).$$

The operator (10) is called the transfer operator of the multivariable system (1)-(2) closed with respect to the ν-th input of the plant.

3. Optimality Condition in Operator Form

Suppose that the controller (2) has been obtained by optimizing the system (1)-(2) (for any $x^{(0)}$) in the sense of the functional

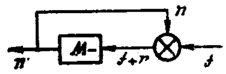

Fig. 1

Fig. 2

$$J=\int_{t_0}^{t_1}(x'H'(t)H(t)x+u'u)\,dt, \tag{11}$$

(where H(t) is an assigned matrix of dimension $n\times n$ such that $Q(t) = H'(t)H(t)$ is positive definite). Optimality [10] signifies that the matrix

$$C(t)=-A(t)B(t), \tag{12}$$

where A(t) is a symmetric matrix of dimension $n\times n$ that satisfies the matrix Riccati equation

$$\dot{A}(t)=-A(t)P(t)-P'(t)A(t)+A(t)B(t)B'(t)A(t)-Q(t),\ A(t_1)=0. \tag{13}$$

THEOREM 1. The transfer operator of a system (1)-(2) that is optimal in the sense of the functional (1) satisfies the equation

$$(I_m+W)^*(I_m+W)=I_m+\check{H}^*\check{H}, \tag{14}$$

where I_m is an m-dimensional identity operator (i.e., $I_m h = h$), and $\check{H} = HMB$.

Formula (14) is called the optimality condition of nonstationary systems in operator form.

THEOREM 2. The transfer operator of a system (1)-(2) that is optimal in the sense of functional (3) and is open-loop with respect to the ν-input of the plant, satisfies the equation

$$(I_1+W_\nu)^*(I_1+W_\nu)=I_1+\check{H}^{\nu*}\check{H}^{\nu}\quad(\nu=1,\ldots,m), \tag{15}$$

where $\check{H}^\nu = H^\nu M_\nu B_{[\nu]}$, $H^{\nu\prime}(t)H^\nu(t) = Q^\nu(t) = Q(t) = Q(t)+\check{C}(t)\check{C}'(t)$.

Theorems 1 and 2 are provided in the Appendix.

4. Control Sensitivity Operator

Suppose that system (1)-(2) is subjected to parametric disturbances. This signifies that the matrices P(t), B(t), and C'(t) can differ from the theoretical (assigned) values by certain matrices $\Delta P(t)$, $\Delta B(t)$, and $\Delta C'(t)$. Parametric disturbances may have various causes (such as inexact description of the plant, inexact realization of the controller parameters, "aging" of the system elements, etc.).

For characterizing the sensitivity of motion of the system (1)-(2) to parametric disturbances, let us introduce a control sensitivity operator. For this purpose let us consider the motions of system (1)-(2) excited by an m-dimensional vector of external disturbances $f(t)\in L_2[t_0, t_1]$:

$$\dot{x}=P(t)x+B(t)(u+f), \tag{16}$$

$$u=C'(t)x. \tag{17}$$

Let us establish a relationship between the control vector and the disturbance vector. For zero initial conditions it follows from (16) that $x = MB(u+f)$, with $u = C^*x = -W(u+f)$, and hence

$$u=-(I_m+W)^{-1}Wf \tag{18}$$

(the symbol (-1) denotes the inverse operator, i.e., $A^{-1}A = AA^{-1} = I_m$).

A block diagram of system (16)-(17) that expresses the relation (18) is shown in Fig. 1.

The circuit represented in Fig. 1 has feedback. The equivalent circuit without feedback is shown in Fig. 2. The operator g of this circuit is selected in such a way that in the absence of parametric disturbances and zero initial conditions we have $u_0 = u$. By virtue of the obvious formula $(I_m+W)^{-1}W = W(I_m+W)^{-1}$, it follows directly from (18) that

$$g=(I_m+W)^{-1}. \tag{19}$$

The operators g and W can be realized by different physical devices; therefore the parametric disturbances of these operators are independent. We shall assume that the operator g is not subjected to parametric disturbances.

Let us consider the difference between the control vectors for the original system and a parametrically disturbed closed-loop system, i.e., $e = u - u_{\alpha}$. For the system represented in Fig. 2 this difference is $e_0 = u_0 - u_{0\alpha}$.

The inequality

$$\|e_0\| > \beta \|e\|, \quad \beta > 1 \tag{20}$$

is called the condition of low sensitivity of a closed-loop system compared to an open-loop system.

The vectors e and e_0 occurring in this condition are related in the case of small parametric disturbances by the following formula (proved in the Appendix):

$$e_0 = (I_m + W)e. \tag{21}$$

With the aid of (21) we obtain a sufficient condition of low sensitivity expressed in terms of the transfer operator of the open-loop system:

$$(I_m + W)^*(I_m + W) > \beta^2 I_m. \tag{22}$$

Indeed, it follows from (22) that $((I_m + W)^*(I_m + W)e, e) \geq \beta^2(e, e)$, or, what amounts to the same $\|(I_m + W)e\| \geq \beta\|e\|$. On the other hand it follows from (21) that $eT\|e_0\| = \|(I_m + W)e\|$, and thus we have proved formula (22).

The operator $(I_m + W)$ occurring in (21) is called the control sensitivity operator. It differs from the operator introduced in [8]. By virtue of its construction, the latter is a sensitivity operator of the plant variables.

5. Properties of Optimal Systems

Property 1. A system (1)-(2) that is optimal in the sense of the functional (11) has low sensitivity to parametric disturbances.

Indeed, by comparing (22) and (14), we conclude that property 1 follows from the inequality $I_m + \check{H}^*\check{H} \geq \beta^2 I_m$, $\beta > 1$, which is easy to prove by taking into account that $(\check{H}^*\check{H}h, h) = \|\check{H}h\|^2 \geq \gamma^2\|h\|^2$, $\gamma > 0$, and hence $((I_m + \check{H}^*\check{H})h, h) \geq \|h\|^2 + \gamma^2\|h\|^2 = \beta^2\|h\|^2$, where $\beta = \sqrt{1 + \gamma^2} > 1$.

Property 2. An optimal system (1)-(2) has low sensitivity with respect to each of the controller outputs.

This property can be expressed by the inequality

$$(I_1 + W_\nu)^*(I_1 + W_\nu) > \beta^2 I_1, \quad (\beta > 1) \quad (\nu = 1, \ldots, m), \tag{23}$$

and its proof follows from (15) and is a repetition of the previous proof.

In going over to property 3, let us consider the system

$$\dot{x} = P(t)x + B(t)(u + f), \quad u = C'(t)x, \quad x^{(0)} = 0, \tag{24}$$

$$\theta = N(t)x, \tag{25}$$

where θ is an m-dimensional vector of controlled variables, f(t) is an m-dimensional vector of external disturbances, N(t) is an assigned matrix of continuous functions of dimension $m \times n$, and the controller matrix $C'(t)$ has been obtained by optimization of the system (1)-(2) in the sense of the functional (11) in which the matrix $H(t) = H_1(t)N(t)$, with $H_1(t)$ being an assigned matrix of dimension $m \times m$ such that $Q_1(t) = H_1'(t)H_1(t)$ is a positive-definite matrix.

Property 3. The processes taking place with respect to the controlled variables in an optimal system subjected to external disturbances satisfy the inequality

$$\int_{t_0}^{t_1} \theta'(t)Q_1(t)\theta(t)\,dt < \int_{t_0}^{t_1} f'(t)f(t)\,dt. \tag{26}$$

APPENDIX

Proof of Theorem 1. Let us consider the operators

$$\Gamma_1 = \dot{A} + AP + P^* A, \quad \Gamma_2 = ABB^* A - H^* H, \tag{A.1}$$

where $\dot{A}$ is an operator generated by the matrix $\dot{A}(t)$, and let us form the products $B^*M^*\Gamma_1 MB$ and $B^*M^*\Gamma_2 MB$. By virtue of (13) we have $\Gamma_1 = \Gamma_2$, and hence

$$B^*M^*\Gamma_1 MB = B^*M^*\Gamma_2 MB. \tag{A.2}$$

Let us express the operators in the left-and right-hand sides of this equation in terms of the operators W and W^*. It follows from (5), (6), and (12) that

$$W = B^* AMB, \quad W^* = B^* M^* AB. \tag{A.3}$$

From (A.1) and (A.3) we directly obtain the right-hand side of (A.2) in the form

$$B^*M^*\Gamma_2 MB = W^* W - \check{H}^* \check{H}, \tag{A.4}$$

where $\check{H} = HMB$.

Now let us write the left-hand side of (A.2) by adding and subtracting from it the oporator $B^*M^*A(I_n + PM)B$,

$$B^*M^*\Gamma_1 MB = B^*M^*(\dot{A} + AP + P^*A)MB + B^*M^*A(I_n + PM)B - B^*M^*A(I_n + PM)B = B^*M^*(\dot{A} + P^*A)MB + B^*M^*A(I_n + PM)B - W^*. \tag{A.5}$$

Below it will be shown that

$$M^*(\dot{A} + P^*A)M + M^*A(I_n + PM) = -AM. \tag{A.6}$$

By subtracting this expression into (A.5), we obtain $B^*M^*\Gamma_1 \; B = -W - W^*$.

By substituting this expression together with (A.4) into (A.2), we obtain the equation $W + W^* + W^*W = \check{H}^*\check{H}$, whence follows (14).

In going over to the proof of (A.6), let us consider

$$\int_t^{t_1} \frac{d}{d\gamma}\left[X'(\gamma, t) A(\gamma) \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma.$$

On the one hand this integral can be written, by taking into account the boundary condition $A(t_1) = 0$, in the form

$$\int_t^{t_1} \frac{d}{d\gamma}\left[X'(\gamma, t) A(\gamma) \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma$$
$$= \left[X'(\gamma, t) A(\gamma) \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] \Bigg|_{\gamma = t}^{\gamma = t_1} = X'(t_1, t) A(t_1) \int_{t_0}^{t_1} X(t_1, \tau)\psi(\tau)\, d\tau - X'(t, t) A(t) \int_{t_0}^{t} X(t, \tau)\psi(\tau)\, d\tau = -AM\psi. \tag{A.7}$$

On the other hand,

$$\int_t^{t_1} \frac{d}{d\gamma}\left[X'(\gamma, t) A(\gamma) \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma = \int_t^{t_1}\left[\frac{dX'(\gamma, t)}{d\gamma} A(\gamma) \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma$$
$$+ \int_t^{t_1}\left[X'(\gamma, t) \frac{dA(\gamma)}{d\gamma} \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma + \int_t^{t_1}\left[X'(\gamma, t) A(\gamma) \frac{d}{d\gamma} \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau \right] d\gamma.$$

Since

$$dX'(\gamma, t)/d\gamma = X'(\gamma, t) P'(\gamma),$$

$$\frac{d}{d\gamma} \int_{t_0}^{\gamma} X(\gamma, \tau)\psi(\tau)\, d\tau = X(\gamma, \gamma)\psi(\gamma) + \int_{t_0}^{\gamma} P(\gamma) X(\gamma, \tau)\psi(\tau)\, d\tau,$$

it follows that

$$\int_t^{t_1} \frac{d}{d\gamma}\left[X'(\gamma, t) A(\gamma) \int_{t_0}^{\gamma} X(\gamma,\tau)\psi(\tau)\,d\tau \right] d\gamma = M^* P^* AM\psi + M^* AM\psi + M^* A[I_n + PM]\psi.$$

By comparing this expression with (A.7), we can see that (A.6) is valid. This completes the proof of Theorem 1.

Proof of Theorem 2. At first let us find a matrix $C^0(t)$ of the controller (9) for which the system (8)-(9) is optimal in the sense of the functional

$$J = \int_{t_0}^{t_1} \{x' [Q(t) + \check{\bar{C}}(t) \check{\bar{C}}'(t)] x + u_\nu^2\}\, dt, \tag{A.8}$$

where $\check{\bar{C}}(t) = -A(t)\check{\bar{B}}(t)$.

The sought vector is specified by the equations

$$C^0(t) = -A^0(t) B(t)_{[\nu]}, \tag{A.9}$$

$$\dot{A}^0(t) = -A^0(t) P_\nu(t) - P_\nu'(t) A^0(t) + A^0(t) B(t)_{[\nu]} B'(t)_{[\nu]} A^0(t) - Q(t) - A(t) \check{\bar{B}}(t) \check{\bar{B}}'(t) A(t), \quad A^0(t_1) = 0. \tag{A.10}$$

By substituting into (A.10) the expression for $P_\nu(t)$, we can easily see that this equation coincides with (13), and hence $A^0(t) = A(t)$. This signifies that $C^0(t) = -A(t)B(t)_{[\nu]}$. From the optimality of system (8)-(9) in the sense of the functional (A.8), and from Theorem 1 and the last equation it follows that (15) is valid. This is true for any ν, and thus we have proved Theorem 2.

Obtaining the Sensitivity Operator. Let us consider the system (16)-(17)

$$\dot{x} = P(t)x + B(t)(u+f), \quad u = C'(t)x.$$

Under parametric disturbances the transfer operator will be

$$W_\alpha = W + \Delta W$$

and the controller output signal

$$u_\alpha = -(I_m + W_\alpha)^{-1} W_\alpha f. \tag{A.11}$$

The difference

$$\begin{aligned} e = u - u_\alpha &= -(I_m+W)^{-1}Wf + (I_m+W_\alpha)^{-1}W_\alpha f \\ &= -(I_m+W_\alpha)^{-1}\{(I_m+W+\Delta W)(I_m+W)^{-1}W - W - \Delta W\}f \\ &= -(I_m+W_\alpha)^{-1}\Delta W\{(I_m+W)^{-1}W - I_m\}f. \end{aligned} \tag{A.12}$$

By taking into account the formula $(I_m + W)^{-1}W - I_m = -(I_m + W)^{-1}$ which is easy to prove, and by multiplying it from the left by $(I_m + W)$, we finally obtain

$$e = (I_m + W_\alpha)^{-1} \Delta W (I_m + W)^{-1} f. \tag{A.13}$$

Now let us consider the open-loop system shown in Fig. 2. Under parametric disturbances we have

$$u_{0\alpha} = W_\alpha g f = -W_\alpha (I_m + W)^{-1} f. \tag{A.14}$$

Let us note that in this equation the operator g remained unchanged, since by construction it has the purpose of ensuring the equivalence of the circuits represented in Figs. 1 and 2 in a parametrically undisturbed state, and after its construction we must therefore "forget" about its origin and assume (as in [8]) that the operator W occurring in g coincides only formally with the operator W of the system (1)-(2).

By constructing the difference

$$e_0 = u_0 - u_{0\alpha} = \Delta W (I_m + W)^{-1} f \tag{A.15}$$

and comparing it with (A.13), we conclude that

$$e_0 = (I_m + W_\alpha) e. \tag{A.16}$$

In the case of small parametric disturbances we have $W_\alpha \approx W$, and therefore (A.16) can be approximately written in the sought form (21).

In conclusion let us note that in deriving the formula (A.16) we used the formal expression $(I_m+W)^{-1}$.

Let us show that this operator exists. Indeed, it follows from Fig. 2 that the vectors f and φ are related by the equation

$$(I_m+W)\varphi=f.$$

In more detailed form, this equation will be written as

$$\varphi(t)-\int_{t_0}^{t} C'(t)X(t,\tau)B(\tau)\varphi(\tau)d\tau=f(t) \tag{A.17}$$

and it constitutes a system of Volterra equations of the second kind. Since the elements of the kernel $K(t, \tau) = C'(t)X(t, \tau)B(\tau)$ and of the vector f(t) are continuous functions, it follows that (A.17) has a unique solution [11], and therefore the operator $(I_m+W)^{-1}$ exists.

Proof of Property 3. Let us find an operator that connects the vectors θ and f in the system (24)-(25):

$$\theta=Nx=NMB[I_m-(I_m+W)^{-1}W]f.$$

By virtue of the formula $I_m-(I_m+W)^{-1}W=(I_m+W)^{-1}$, we obtain

$$\theta=NMB(I_m+W)^{-1}f. \tag{A.18}$$

It is easy to see that

$$(H_1\theta, H_1\theta)=(\check{H}(I_m+W)^{-1}f,\ \check{H}(I_m+W)^{-1}f)=(\check{H}^*\check{H}(I_m+W)^{-1}f,\ (I_m+W)^{-1}f), \tag{A.19}$$

where $\check{H} = H_1NMB$.

Taking into account the optimality of the controller of the system under consideration, let us eliminate $\check{H}^*\check{H}$ from (A.19) with the aid of (14). Hence we obtain the formula

$$(((I_m+W)^*(I_m+W)-I_m)(I_m+W)^{-1}f,\ (I_m+W)^{-1}f)=(f', f)-((I_m+W)^{-1}f,\ (I_m+W)^{-1}f),$$

whence it follows with the use of (A.19) and of the inequality $((I_m+W)^{-1}f,\ (I_m+W)^{-1}f)\geq 0$ that $(H_1\theta, H_1\theta)\leq(f', f)$. Thus we have proved the property 3.

The author expresses his gratitude to A. P. Khromov for his assistance and useful advice.

LITERATURE CITED

1. A. M. Letov, "Analytic design of controllers. I-IV," Avtom. Telemekh., No. 4, 436-441; No. 5, 561-568; No. 6, 661-665 (1960); No. 4, 425-435 (1961).
2. A. M. Letov, "Theory of optimal control," in: Proceedings Second IFAC Congress. Optimal Systems. Statistical Methods, Nauka (1965), pp. 7-38.
3. R. E. Kalman, "Contributions to the theory of optimal control," Bol. Soc. Mat. Mex., Ser. 2, 5, No. 1, 102-119 (1960).
4. A. G. Aleksandrov, "Frequency properties of optimal linear systems," Avtom. Telemekh., No. 9, 176-181 (1969).
5. A. G. Aleksandrov, "Analytic design of transfer matrices of controllers on the basis of frequency performance criteria. II," Avtom. Telemekh., No. 2, 17-29 (1972).
6. A. G. Aleksandrov, "Properties of analytically designed systems," Avtom. Telemekh., No. 10, 5-11 (1975).
7. R. E. Kalman, "When is a linear system optimal?" Proc. ASME, Ser. D, No. 1 (1964).
8. W. Perkis and D. Krus, "Sensitivity operators for linear systems with variable parameters," in: Control System Sensitivity [Russian translation], Nauka (1968), pp. 45-51.
9. N. N. Krasovskii, Theory of Control of Motion [in Russian], Nauka (1968).
10. Ya. N. Roitenberg, Automatic Control [in Russian], Nauka (1971).
11. S. G. Mikhlin, Lectures on Linear Integral Equations [in Russian], Fizmatgiz (1959).