FINITE-FREQUENCY CRITERIA FOR STABILITY OF SYSTEMS WITH UNDETERMINED PARAMETERS

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Constructive necessary and sufficient conditions are obtained for the stability of linear systems whose plants are given by a finite number of values of their frequency response.

1. Introduction

The problem of the analysis of the stability of systems of automatic control with the use of experimental data is of great interest. The criterion of Nyquist [1, 2] lies at the foundation of such analysis. Its use presupposes a knowledge of the obtained experimental frequency response of the open system, in general, on the entire range of frequencies from zero to infinity. In addition, it is necessary to have information concerning the number of poles of the transfer function of the open system which have positive real part. It is possible to reduce the number of experiments if one determines (identifies) the parameters of the transfer function of the open system on the basis of trials for a small finite number of frequencies, and then calculates the frequency response for frequencies $\omega \in [0, \infty]$. The question arises in this connection as to whether it is possible to find direct (unidentified) criteria of stability which use only results of trials for a finite number of frequencies without identification of parameters and construction of the frequency response on the entire range of frequencies.

In what follows, we obtain two such criteria for the case where the controller parameters are known and the plant is given by a finite number of points of its frequency response. These criteria are based on the construction of certain optimization functionals (in particular, by means of the solution of the inverse optimal control problem [3]) and the solution of the problem of the analytical construction of optimal controllers [4] for plants given by their frequency response [5]. If the optimal controller obtained coincides with the given one, this means that the system under study is stable.

2. Statement of the Problem

We consider two kinds of models of control systems.

First kind consists of mopdels described by the equations

$$\dot{x}=Px+bu,\ x(t_0)=x^{(0)}, \tag{1}$$

$$u=c^{*\prime}x, \tag{2}$$

where x(t) is an n-dimensional vector of state variables of the plant (1) which are directly measurable, u(t) is the control (the result of the controller (2)), c* is an n-dimensional numerical vector (the prime denotes transposition); the numbers (parameters) making up the matrix P and the n-dimensinal vector b are unknown.

We will suppose that the plant (1) is completely controllable and observable with respect to the signal u. This means that

$$\det\|b, Pb, \ldots, P^{n-1}b\|\neq 0,\ \det\|c^*, Pc^*, \ldots, P^{n-1}c^*\|\neq 0. \tag{3}$$

We suppose that the plant (1) is asymptotically stable, or if it is unstable, that the n-dimensional vector $c^{(0)}$ of the controller

$$u=c^{(0)\prime}x, \tag{4}$$

is known so that the system (1), (4) is asymptotically stable. This property of the plant allows us to obtain experimentally its amplitude-frequency characteristic $a_i(\omega)$ and its phase-

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frequency characteristic $\varphi_i(\omega)$ $(i=\overline{1, n})$, which determine the steady-state movements by the outputs of the plant $x_i(t)=a_i(\omega)\sin[\omega t+\varphi_i(\omega)]$ $(i=\overline{1, n})$ when u = sin ωt is applied to its input. Using these characteristics, we formulate the frequency response (vector) β(jω) of the plant with components $\beta_i(\omega)=a_i(\omega)e^{j\varphi_i(\omega)}$ $(i=\overline{1, n})$. It is related to the parameters of the plant by the relations

$$\beta(j\omega)=(E_n j\omega-P)^{-1}b, \tag{5}$$

where E_n is the n × n identity matrix.

We will say that the plant is given by a finite number of values of its frequency response if the values of β(jω) are known for some set of distinct test frequencies $\omega_1, \ldots, \omega_n$.

The second kind consists of models described by equations of "input–output" form:

$$y^{(n)}+d_{n-1}y^{(n-1)}+\ldots+d_1\dot{y}+d_0y=k_\gamma u^{(\gamma)}+\ldots+k_1u+k_0u \quad (\gamma<n), \tag{6}$$

$$u^{(n)}+g_{n-1}^{*}u^{(n-1)}+\ldots+g_1^{*}\dot{u}+g_0^{*}u=r_0^{*}y+r_1^{*}y+\ldots+r_{n-1}^{*}y^{(n-1)}, \tag{7}$$

where y(t) is the output variable of the plant (6), where $y^{(i)}$ and $u^{(i)}$ are the i-th derivatives, the parameter d_i, $(i=\overline{0, n-1})$ and k_j $(j=\overline{0, \gamma})$ of the plant are unknown, and the coefficients g_i^*, r_j^* $(i=\overline{0, n};\ j=\overline{0, n})$ of the controller (7) are given.

We will assume that the transfer function of the plant (6)

$$h(s)=\frac{k_\gamma s^\gamma+\ldots+k_1s+k_0}{s^n+d_{n-1}s^{n-1}+\ldots+d_1s+d_0}=\frac{k(s)}{d(s)} \tag{8}$$

is such that none of the roots of the polynomial k(s) coincides with the roots of the polynomial d(s).

The plant (6) is asymptotically stable, or if it is unstable, it is possible to find a controller

$$\sum_{i=0}^{n} g_i^{(0)}u^{(i)}=\sum_{j=0}^{n-1} r_j^{(0)}y^{(j)} \quad (g_n^{(0)}=1), \tag{9}$$

such that the system (6), (9) is asymptotically stable [6].

Under this assumption, it is possible to obtain the values $h(j\omega_k)=a_y(\omega_k)e^{j\varphi_y(\omega_k)}$ $(k=\overline{1, 2n})$ of the frequency response of the plant experimentally.

The problem consists in finding necessary and sufficient conditions (criteria) for the stability of systems of the first and second kind whose plant are given by a finite number of values of their frequency response $\beta(j\omega_k)$ $(k=\overline{1, n})$ and $h(j\omega_k)$ $(k=\overline{1, 2n})$, respectively. In addition, $\omega_i \neq \omega_k$ for $i \neq k$, where $i, k=\overline{1, n}$ or $i, k=\overline{1, 2n}$.

3. First Stability Criterion

First of all, we obtain a necesary condition for the stability of the system (1), (2). In this connection, we consider the inverse optimal control problem [3]. This problem consists in finding a positive definite functional

$$J=\int_0^\infty [x'Q^{(1)}x+2(l'x)u+u^2]dt, \tag{10}$$

($Q^{(1)}$ is an n × n matrix and ℓ an n-dimensional vector) with respect to which the control (2) is optimal.

If the system (1), (2) is asymptotically stable and satisfies (3), then there always exist a matrix $Q^{(1)} > 0$ and vector ℓ such that [3]

$$Q_l=Q^{(1)}-ll'>0. \tag{11}$$

Since the matrix $Q^{(1)}$ and the vector ℓ constitute a solution of the inverse optimal control problem, they satisfy the identity [3]

$$[1-(c^{*\prime}+l')(-E_nj\omega-P+bl')^{-1}b][1-(c^{*\prime}+l')(E_nj\omega-P+bl')^{-1}b]=$$
$$=1+b'(-E_nj\omega-P+bl')^{-1\prime}Q_l(E_nj\omega-P+bl')^{-1}b,\ 0\leqslant\omega\leqslant\infty. \tag{12}$$

We use this identity and the inequality (11) to find $Q^{(1)}$ and ℓ for the given $\beta(j\omega_k)$ $(k=\overline{1,n})$ and c^*.

Using the relation

$$(E_nj\omega-P+bl')^{-1}b=(E_nj\omega-P)^{-1}b[1+l'(E_nj\omega-P)^{-1}b]^{-1}, \tag{13}$$

from [5], and taking (5) into account, we can write (12) in the form

$$\{1-(c^{*\prime}+l')\beta(-j\omega)[1+l'\beta(-j\omega)]^{-1}\}\{1-(c^{*\prime}+l')\beta(j\omega)[1+$$
$$+l'\beta(j\omega)]^{-1}\}=1+\beta'(-j\omega)[1+l'\beta(-j\omega)]^{-1}Q_l\beta(j\omega)[1+l'\beta(j\omega)]^{-1}. \tag{14}$$

Finally, after some simple transformations, we get

$$l'\beta(-j\omega)+l'\beta(j\omega)+\beta'(-j\omega)Q^{(1)}\beta(j\omega)=\gamma(\omega),\ 0\leqslant\omega\leqslant\infty, \tag{15}$$

where

$$\gamma(\omega)=-c^{*\prime}\beta(-j\omega)-c^{*\prime}\beta(j\omega)+\beta'(-j\omega)c^*c^{*\prime}\beta(j\omega). \tag{16}$$

Putting $\omega=\omega_k$ $(k=\overline{1,n})$ (in (15) and (16), we get the system of n algebraic equations

$$l'\beta(-j\omega_k)+l'\beta(j\omega_k)+\beta'(-j\omega_k)Q^{(1)}\beta(j\omega_k)=\gamma(\omega_k)\ \ (k=\overline{1,n}). \tag{17}$$

<u>Proposition 1</u>. The set of matrices $Q^{(1)}$ and vectors ℓ which satisfy the system (17) coincides with the set of those matrices and vectors which satisfy (15). In other words, (17) is necessary and sufficient for (15).

The proof of this proposition is simply a repetition of the proof of the analogous statement in [5].

For convenience, we write (17) in matrix form. In this connection we introduce the notation

$$M=2\begin{Vmatrix} a_1(\omega_1)\cos\varphi_1(\omega_1), \dots, & a_n(\omega_1)\cos\varphi_n(\omega_1) \\ a_1(\omega_2)\cos\varphi_1(\omega_2), \dots, & a_n(\omega_2)\cos\varphi_n(\omega_2) \\ \vdots & \vdots \\ a_1(\omega_n)\cos\varphi_1(\omega_n), \dots, & a_n(\omega_n)\cos\varphi_n(\omega_n)\end{Vmatrix},\quad \gamma=\begin{Vmatrix}\gamma(\omega_1)\\ \gamma(\omega_2)\\ \vdots \\ \gamma(\omega_n)\end{Vmatrix}, \tag{18}$$

$$N_-=\|\beta(-j\omega_1),\dots,\beta(-j\omega_n)\|,\ N_+=\|\beta(j\omega_1),\dots,\beta(j\omega_n)\|, \tag{19}$$

and $D\{A\}$ is the column vector whose components are the respective elements of the diagonal of the square matrix A.

In this notation (17) takes the form

$$Ml+D\{N_-'Q^{(1)}N_+\}=\gamma, \tag{20}$$

where the n-dimensional vector

$$\gamma=-Mc^*+D\{N_-'c^*c^{*\prime}N_+\}. \tag{21}$$

is known.

If

$$\det M\neq 0, \tag{22}$$

then we find from (20) that

$$l=M^{-1}[\gamma-D\{N_-'Q^{(1)}N_+\}]. \tag{23}$$

Substituting this expression into (11), we get

$$Q^{(1)}-M^{-1}[\gamma-D\{N_-'Q^{(1)}N_+\}][\gamma-D\{N_-'Q^{(1)}N_+\}]'M^{-1\prime}>0. \tag{24}$$

If the plant (1) is asymptotically stable, then, as is shown in [7], the inequality (22) is fulfilled. If the plant is stable, beginning with (12), we must replace P by $P^{(0)}=P+bc^{(0)\prime}$, and $\beta(j\omega)$ by $\beta^{(0)}(j\omega)=\beta(j\omega)[1-c^{(0)\prime}\beta(j\omega)]$; then the matrix for the vector ℓ in (20) will again be nonsingular.

If the system (1), (2) is asymptotically stable, then there always exists [3] a positive definite matrix $Q^{(1)}$ which satisfies (24) and thus this inequality is a necessary condition of stability. However, it can also happen that one obtains a $Q^{(1)} > 0$ from (24) for an unstable system (1), (2). The inequality (24) will be sufficient for the asymptotic stability of the system under consideration if as a result of solving the problem of optimizing the functional (10), in which $Q^{(1)}$ and ℓ are obtained from (24) and (23), we get

$$c=c^*. \tag{25}$$

Passing to the consideration of the problem of the analytic construction of optimal controllers for plants given by their frequency response, we write the functional (10) in the form

$$J=\int_0^\infty [x'(Q^{(1)}-ll')x+(u+l'x)^2]dt \tag{26}$$

and introduce the "new" control

$$u_l=u+l'x. \tag{27}$$

Then, if we take into account the notation (11), Eq. (1) and the functional (26) take the form

$$\dot{x}=P_lx+bu_l,\quad x(t_0)=x^{(0)}, \tag{28}$$

$$J=\int_0^\infty (x'Q_lx+u_l^2)dt, \tag{29}$$

where

$$P_l=P-bl'. \tag{30}$$

The "plant" (28) is given by the frequency response

$$\beta_l(j\omega_k)=(E_nj\omega_k-P_l)^{-1}b=(E_nj\omega_k-P+bl')^{-1}b=\beta(j\omega_k)[1+l'\beta(j\omega_k)]^{-1}\quad (k=\overline{1,n}). \tag{31}$$

It is completely controllable since the controllability of the pair (P_ℓ, b) follows from the controllability of the pair (P, b) [8]. It is possible to find a control

$$u_l=(c^{(0)}+l)'x=c_l^{(0)\prime}x, \tag{32}$$

for it such that the system (28), (32) is asymptotically stable.

The optimal control

$$u_l=c_l'x, \tag{33}$$

for which the functional (29) is minimized by motions of the system (28), (33) actuated by an arbitrary vector $x^{(0)}$ can be found by solving the recursive system [5]

$$-2\,\mathrm{Re}\{c_l^{(\alpha+1)\prime}\beta_l(j\omega_k)[1-c_l^{(\alpha)\prime}\beta_l(-j\omega_k)]\}= \tag{34}$$
$$=\beta_l'(-j\omega_k)[Q_l+c_l^{(\alpha)}c_l^{(\alpha)\prime}]\beta_l(j\omega_k)\quad (\alpha=0,1,\ldots,\ k=\overline{1,n}).$$

The properties of the "plant" (28) guarantee [5] the convergence of the sequence $c_\ell^{(\alpha)}$ of vectors, so that

$$\lim_{\alpha\to\infty} c_l^{(\alpha)}=c_l. \tag{35}$$

<u>Proposition 2 (stability criterion)</u>. For the asymptotic stability of the system (1), (2) it is necessary that there exist a matrix $Q^{(1)} > 0$ which satisfies (24), and it is sufficient that $Q^{(1)}$ and the vector ℓ found from (24) and (23) be such that the vectors $c_\ell^{(\alpha)}$ which are solutions of the system (34) have the property that

$$\lim_{\alpha\to\infty} c_l^{(\alpha)}=c^*+l. \tag{36}$$

The practical application of this criterion is difficult because one must find $Q^{(1)}$ from the nonlinear inequality (24). We note that in some cases this inequality is considerably simplified. Thus if the condition

$$[1-c^{*\prime}\beta(-j\omega)][1-c^{*\prime}\beta(j\omega)]>1,\quad 0\leq\omega\leq\infty, \tag{37}$$

is fulfilled, then $\ell = 0$ [3], and we can find only $Q^{(1)} > 0$ from the equality

$$D\{N_-'Q^{(1)}N_+\}=\gamma. \tag{38}$$

The condition (37) is fulfilled [9] if the vector c^* is obtained as a result of solving the problem of the analytic construction of an optimal control or the analytic construction of an optimal control according to the criterion in the extended articles [10].

4. Second Stability Criterion

We introduce the functional

$$J=\int_0^\infty [\varepsilon^2 x'Q^{(2)}x+(u-c^{*\prime}x)^2]dt, \tag{39}$$

where $Q^{(2)}$ is an arbitrary given positive definite $n \times n$ matrix, and ε is some number (undetermined parameter). We study the dependence on ε of the control

$$u=c'(\varepsilon)x, \tag{40}$$

for which the functional (39) achieves its smallest value under movements of the system (1), (40).

Proposition 3. If the system (1), (2) is asymptotically stable, then

$$c(\varepsilon)\to c^*, \tag{41}$$

as $\varepsilon \to 0$, but if this system is unstable, then

$$c(\varepsilon)\to c^{**}\neq c^*. \tag{42}$$

as $\varepsilon \to 0$.

The proof of this proposition is given in the appendix.

To calculate $c(\varepsilon)$, we introduce the "new" control

$$u_l=u-c^{*\prime}x, \tag{43}$$

then (1) and (39) take the form (28) and (29), where

$$l=-c^*,\ P_l=P+bc^{*\prime},\ Q_l=\varepsilon^2 Q^{(2)}. \tag{44}$$

Solving the recursive system (34) for these values, we get

$$c_l(\varepsilon)=\lim c_l^{(\alpha)}(\varepsilon). \tag{45}$$

(for each fixed ε).

According to (43),

$$c_l(\varepsilon)=c(\varepsilon)-c^* \tag{46}$$

and consequently, if the system (1), (2) is asymptotically stable, then $c_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$, and if it is unstable, then $c_\ell(\varepsilon) \to c^{**} - c^* \neq 0$.

Proposition 4 (stability criterion). The system (1), (2) is asymptotically stable if and only if the vector $c_\ell(\varepsilon)$ produced by the system (34) for various values of ε in (44) have the property that

$$c_l(\varepsilon)\to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{47}$$

The necessity of (47) follows from (41), and the sufficiency from the optimality of (43).

The relation (47) means that, for sufficiently small ε, the vector $c_\ell(\varepsilon)$ must be sufficiently small.

To use the criterion in practice, one must define what is meant by sufficiently small. In connection with this remark, it is not difficult to show that

$$c_l(\varepsilon)\approx\varepsilon^2 T \tag{48}$$

(where T is some vector which does not depend on ε), which holds for sufficiently small ε. In other words, if c_ℓ, the solution to the system (34), changes in proportion to ε^2, then this means that the parameter ε is sufficiently small.

We define the smallness of c_ℓ in terms of its relation to the unknown vector c^*, and we say that c_ℓ is sufficiently small if

$$|c_{ii}|\ll|c_i^*|\quad (i=\overline{1,n}). \tag{49}$$

5. Application of the Criterion to Systems of the Second Kind

We show that the system (6), (7) can be reduced to the form (1), (2), and that we can construct the vector $\beta(j\omega_k)$ $(k = \overline{1,\ 2n})$ from the function $h(j\omega_k)$.

We write the control equation (7) in the form of two equations:

$$u^{(n)}+\mu_{n-1}u^{(n-1)}+\ldots+\mu_1\dot{u}+\mu_0 u=\breve{u}, \tag{50}$$

$$\breve{u}=r_0^* y+\ldots+r_{n-1}^* y^{(n-1)}+(-g_0^*+\mu_0)u+\ldots+(-g_{n-1}^*+\mu_{n-1})u^{(n-1)}, \tag{51}$$

where μ_i $(i = \overline{0,\ n-1})$ are arbitrarily given positive numbers such that the roots of the equation $\mu(s) = s^n + \mu_{n-1}s^{n-1} + \ldots + \mu_1 s + \mu_0 = 0$ have negative real parts.

Putting (6) and (50) together, we get the new "plant"

$$d(s)y=k(s)u,\quad \mu(s)u=\breve{u}. \tag{52}$$

We introduce the notation

$$y=\breve{x}_1,\ \dot{y}=\breve{x}_2,\ldots,y^{(n-1)}=\breve{x}_n,\ u=\breve{x}_{n+1},\ \dot{u}=\breve{x}_{n+2},\ldots,u^{(n-1)}=\breve{x}_{2n} \tag{53}$$

$$r_{i-1}^*=\breve{c}_i^*,\ (-g_{i-1}^*+\mu_{i-1})=\breve{c}_{n+i}^*\quad (i=\overline{1,n}). \tag{54}$$

Then the Eqs. (52), (51) are equivalent to (6), (7) and take the form

$$\dot{\breve{x}}=\breve{P}\breve{x}+\breve{b}\breve{u}, \tag{55}$$

$$\breve{u}=\breve{c}^{*\prime}\breve{x}, \tag{56}$$

where

$$\breve{x}=\begin{Vmatrix}\breve{x}_1\\ \breve{x}_2\\ \vdots\\ \breve{x}_n\\ \breve{x}_{n+1}\\ \breve{x}_{n+2}\\ \vdots\\ \breve{x}_{2n}\end{Vmatrix}, \tag{57}$$

$$\breve{P}=\begin{Vmatrix}0 & 1 & 0,\ldots, & 0 & 0 & 0 & 0,\ldots, & 0\\ 0 & 0 & 1,\ldots, & 0 & 0 & 0 & 0,\ldots, & 0\\ \vdots & & & & \vdots & & & \\ -d_0 & -d_1 & -d_2,\ldots & -d_{n-1}, & k_0, & k_1, & k_2,\ldots, & k_{n-1}\\ 0 & 0 & 0,\ldots, & 0, & 0 & 1 & 0,\ldots, & 0\\ 0 & 0 & 0,\ldots, & 0, & 0 & 0 & 1,\ldots, & 0\\ \vdots & & & & & & & \\ 0 & 0 & 0,\ldots, & 0, & -\mu_0 & -\mu_1 & -\mu_2,\ldots, & -\mu_{n-1}\end{Vmatrix},$$

$$\breve{b}=\begin{Vmatrix}0\\0\\ \vdots\\0\\0\\0\\ \vdots\\1\end{Vmatrix},\quad \breve{c}^*=\begin{Vmatrix}\breve{c}_1^*\\ \breve{c}_2^*\\ \vdots\\ \breve{c}_n^*\\ \breve{c}_{n+1}^*\\ \breve{c}_{n+2}^*\\ \vdots\\ \breve{c}_{2n}^*\end{Vmatrix}.$$

We now express the transfer matrix $\breve{\beta}(j\omega_k)$ $(k = \overline{1,\ 2n})$ of the "plant" (55) in terms of the known values $h(j\omega_k)$ $(k = \overline{1,\ 2n})$ of the transfer function.

By definition

$$\breve{\beta}(j\omega)=(E_{2n}j\omega-\breve{P})^{-1}\breve{b}. \tag{58}$$

If we take into account the structure of the matrix $\check{P}$, we find that

$$(E_{2n}j\omega-\check{P})^{-1}=\left\|\begin{array}{c|c}T_1 & T_2\\ \hline 0 & T_4\end{array}\right\|^{-1}=\left\|\begin{array}{c|c}T_1^{-1} & -T_1^{-1}T_2T_4^{-1}\\ \hline 0 & T_4^{-1}\end{array}\right\|, \tag{59}$$

where

$$T_1=\left\|\begin{array}{llll}j\omega & -1 & 0,\ldots,0 & \\ 0 & j\omega & -1,\ldots,0 & \\ \vdots & & & \\ d_0 & d_1 & d_2,\ldots, & j\omega+d_{n-1}\end{array}\right\|, \tag{60}$$

$$T_2=\left\|\begin{array}{llll}0 & 0 & 0,\ldots, & 0\\ 0 & 0 & 0,\ldots, & 0\\ \vdots & & & \\ -k_0 & -k_1 & -k_2,\ldots, & -k_{n-1}\end{array}\right\|,$$

$$T_4=\left\|\begin{array}{llll}j\omega & -1 & 0,\ldots, & 0\\ 0 & j\omega & -1,\ldots, & 0\\ \vdots & & & \\ \mu_0 & \mu_1 & \mu_2,\ldots, & j\omega+\mu_{n-1}\end{array}\right\|.$$

Since the vector $\check{\beta}(j\omega)$ is the last column of the matrices $T_1^{-1}T_2T_4^{-1}$ and T_4^{-1}, we have that

$$\check{\beta}(j\omega_k)=\|h^{(1)}(j\omega_k),\ (j\omega_k)h^{(1)}(j\omega_k),\ldots$$
$$\ldots,(j\omega_k)^{n-1}h^{(1)}(j\omega_k),\ \mu^{-1}(j\omega_k),\ldots,(j\omega_k)^{n-1}\mu^{-1}(j\omega_k)\|\quad (k=\overline{1,2n}), \tag{61}$$

where

$$h^{(1)}(j\omega_k)=h(j\omega_k)\mu^{-1}(j\omega_k)\quad (k=\overline{1,2n}). \tag{62}$$

Thus we have the "plant" (55) given by the frequency response (62). It is asymptotically stable if the plant (6) is. This follows from the equality $\det(E_{2n}s-\check{P})=\det(E_ns-T_1)\times\det(E_ns-T_4)=d(s)\mu(s)$. If the plant (6) is unstable, then using (9) it is possible to find a stabilizing control $u=c^{(0)\prime}x$ for the plant (55).

From the components of the vector $\check{\beta}(j\omega)$ in (61), (62) it follows that the "plant" (55) is completely controllable only if the roots of the polynomial $\mu(s)$ do not coincide with the roots of the numerator $k(s)$ of the transfer function of the plant (6). The "control law" (56) is observable only if the numerator and denominator of the transfer function

$$\check{c}^{*\prime}\beta(s)=\frac{\left[\sum_{i=1}^{n}c_i^*s^{(i-1)}\right]k(s)+\left[\sum_{i=n+1}^{2n}c_i^*s^{(n-i)}\right]d(s)}{d(s)\mu(s)} \tag{63}$$

do not have common roots. Under these conditions, it is possible to apply the first and second stability criteria for the system (55), (56), which is equivalent (in the sense of the stability property under consideration) to the system (6), (7).

6. Examples

1°. We apply the first criterion to analyze the stability of the system

$$\dot{x}_1=x_2,\quad \dot{x}_2=-d_0x_1-d_1x_2+bu,\quad u=c_1^*x_1+c_2^*x_2. \tag{64}$$

Concerning the parameters of the plant we know only that

$$d_0>0,\quad d_1>0,\quad b\neq 0. \tag{65}$$

As a result of trials with this plant under harmonic actions with frequencies $\omega_1=10$, $\omega_2=20$ we obtain the steady-state amplitudes and phase shifts

$$a_1(\omega_1)=3.3\cdot 10^{-2},\ \varphi(\omega_1)=-72^\circ,\ a_1(\omega_2)=0.9\cdot 10^{-2},\ \varphi_1(\omega_2)=-128^\circ. \tag{66}$$

in the variable x_1 at these frequencies.

It follows immediately from the equations of the plant that, for x_2,

$$a_2(\omega_1)=\omega_1 a_1(\omega_1)=3.3\cdot10^{-2},\quad \varphi_2(\omega_1)=\frac{\pi}{2}+\varphi_1(\omega_1)=18°, \tag{67}$$

$$a_2(\omega_2)=\omega_2 a_1(\omega_2)=2.7\cdot10^{-2},\quad \varphi_2(\omega_2)=\frac{\pi}{2}+\varphi_1(\omega_2)=-38°. \tag{68}$$

The control parameters of the system (64) are

$$c_1^*=-2.7\cdot10^3,\quad c_2^*=-1.6\cdot10^2. \tag{69}$$

Passing to an analysis of the stability, we put $\ell_1 = \ell_2 = 0$, and we seek the parameters q_{11} and q_{22} of the analog matrix $Q^{(1)}$ from (38). Using (18) and (19), we construct the matrices

$$N_+=\begin{Vmatrix} a_1(\omega_1)e^{j\varphi_1(\omega_1)}, & a_1(\omega_2)e^{j\varphi_1(\omega_2)} \\ \omega_1 a_1(\omega_1)e^{j[\varphi_1(\omega_1)+\pi/2]}, & \omega_2 a_1(\omega_2)e^{j[\varphi_1(\omega_2)+\pi/2]} \end{Vmatrix}, \tag{70}$$

$$M=2\begin{Vmatrix} a_1(\omega_1)\cos\varphi_1(\omega_1), & \omega_1 a_1(\omega_1)\cos[\varphi_1(\omega_1)+\pi/2] \\ a_1(\omega_2)\cos\varphi_1(\omega_2), & \omega_2 a_1(\omega_2)\cos[\varphi_1(\omega_2)+\pi/2] \end{Vmatrix}. \tag{71}$$

Equation (38) has the form

$$q_{11}+q_{22}\omega_k^2=\frac{1}{a_1^2(\omega_k)}\Big[-2c_1^*a_1(\omega_k)\cos\varphi_1(\omega_k)- -2c_2^*\omega_k a_1(\omega_k)\cos\Big[\frac{\pi}{2}+\varphi_1(\omega_k)\Big]+(c_1^{*2}+c_2^{*2}\omega_k^2)a_1^2(\omega_k)\Big]\quad (k=1,2). \tag{72}$$

Solving this system for the values (66)-(69), we get

$$q_{11}=8\cdot10^6,\quad q_{22}=3\cdot10^4. \tag{73}$$

Hence it follows that the necessary condition for stability is fulfilled. To verify the sufficient condition, we must solve the problem of minimizing the functional

$$J=\int_0^\infty (8\cdot10^6x_1^2+3\cdot10^4x_2^2+u^2)\,dt \tag{74}$$

under movements of the plant (64) given by the frequency response (66)-(68).

The solution to this problem is obtained in [5] (p. 166), where it is shown that $\lim_{\alpha\to\infty} c_1^{(\alpha)}=-2.7\cdot10^3$, $\lim_{\alpha\to\infty} c_2^{(\alpha)}=-1.6\cdot10^2$, which means that the system (64) is asymptotically stable.

2°. We apply the second criterion to analyze the stability of the system

$$\dot{x}_1=-d_0x_1+bu,\quad u=c_1^*x_1, \tag{75}$$

for which the parameters of the plant (d_0 and b) are unknown. It is only known that $b \neq 0$. In addition, it is known that the control $u = c_1^{(0)}x$, $c_1^{(0)} = -3$, for which the plant of the system (75), closed by this control, is asymptotically stable. In this case it is possible to determine the values of the amplitude-frequency and phase-frequency characteristics of the plant experimentally. Suppose that for $\omega_1 = 2.25$ we have that

$$a_1(\omega_1)=\frac{1}{\omega_1}=\frac{1}{2.25},\quad \varphi_1(\omega_1)=-\frac{\pi}{2}. \tag{76}$$

We will study two situations. In the first, $c_1^* = -2$, and in the second $c_1^* = 2$.

Passing to the analysis of stability for $c_1^* = -2$, we write (34):

$$-2c_1^{(\alpha+1)}a_l(\omega_1)[\cos\varphi_l(\omega_1)-c_1^{(\alpha)}a_l(\omega_1)]=a_l^2(\omega_1)[e^2q_{11}+c_1^{(\alpha)2}] \quad (\alpha=0,1,2,\ldots), \tag{77}$$

where, according to (31) and (44),

$$a_l(\omega_1)e^{j\varphi_l(\omega_1)}=a_1(\omega_1)e^{j\varphi_1(\omega_1)}[1-c_1^*a_1(\omega_1)e^{j\varphi_1(\omega_1)}]^{-1}. \tag{78}$$

Taking (76) into account, we get that

$$a_l(\omega_1)=\frac{1}{\sqrt{\omega_1^2+c_1^{*2}}}=\frac{1}{3},\quad \cos\varphi_l(\omega_1)=\frac{-c_1^*}{\sqrt{\omega_1^2+c_1^{*2}}}=\frac{2}{3}. \tag{79}$$

From (77) it follows that

$$c_l^{(\alpha+1)} = -\frac{a_l(\omega_1)[\varepsilon^2 q_{11} + c_l^{(\alpha)2}]}{2[\cos\varphi_l(\omega_1) - c_l a_l(\omega_1)]} \quad (\alpha=0, 1, 2, \ldots). \tag{80}$$

From (32) we find that

$$c_l^{(0)} = c^{(0)} - c_1^* = -1. \tag{81}$$

Then, using (80) and putting $\varepsilon^2 q_{11} = 0.2$, we get

$$c_l^{(1)} = 0.2; \quad c_l^{(2)} = -0.054; \quad c_l^{(3)} = -0.049. \tag{82}$$

We put

$$\varepsilon^2 q_{11} = 0.1. \tag{83}$$

As a result, we find that

$$c_l^{(1)} = -0.183; \quad c_l^{(2)} = -0.03; \quad c_l^{(3)} = -0.0248. \tag{84}$$

From the inequality $|-2| \gg |-0.049|$ and the fact that c_ℓ is proportional to ε^2, we can conclude that (75) is asymptotically stable for $c_1^* = -2$.

Now we let $c_1^* = 2$. In this case in (79) we must put $\cos\ \varphi_l(\omega_1) = -^2/_3$, and in (81) $c_\ell^{(0)} = c^{(0)} - c_1^* = -5$.

For $\varepsilon^2 q_{11} = 0.2$, we get by (80) that

$$c_l^{(1)} = -4.2; \quad c_l^{(2)} = -4.054; \quad c_l^{(3)} = -4.05.$$

For $\varepsilon^2 q_{11} = 0.1$, we have

$$c_l^{(1)} = -4.18; \quad c_l^{(2)} = -4.01; \quad c_l^{(3)} = -4.005.$$

Thus the system (75) is unstable for $c_1^* = 2$.

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APPENDIX

Proof of Proposition 3

The vector $c(\varepsilon)$, which is the solution to the problem of minimizing the functional (39) over the motions of the system (1), (40), satisfies the identity

$$[1-c'(\varepsilon)\beta(-s)][1-c'(\varepsilon)\beta(s)] = [1-c^{*\prime}\beta(-s)][1-c^{*\prime}\beta(s)] + \varepsilon^2\beta'(-s)Q^{(2)}\beta(s), \quad s=j\omega, \quad 0\leq\omega<\infty. \tag{A.1}$$

It is not difficult to obtain this identity from (15) and (16) by replacing c^* by $c(\varepsilon)$ and putting

$$l = -c^*, \quad Q^{(1)} = Q^{(2)} + c^* c^{*\prime}. \tag{A.2}$$

We now investigate the dependence of $c(\varepsilon)$ on ε. To do this, we represent the components of the vector

$$\beta_i(s) = \tilde\beta_i(s)/d(s) \quad (i=\overline{1, n}), \tag{A.3}$$

where

$$d(s) = \det(E_n s - P), \tag{A.4}$$

and we have put

$$\tilde\beta(s) = \|\tilde\beta_1(s), \ldots, \tilde\beta_n(s)\|. \tag{A.5}$$

Then (A.1) can be written in the form

$$D_\varepsilon(-s)D_\varepsilon(s) = D^*(-s)D^*(s) + \varepsilon^2\tilde\beta'(-s)Q^{(2)}\tilde\beta(s). \tag{A.6}$$

Here

$$D_\varepsilon(s) = d(s) - c'(\varepsilon)\tilde\beta(s), \quad D^*(s) = d(s) - c^{*\prime}\tilde\beta(s), \tag{A.7}$$

where $D_\varepsilon(s)$ is the characteristic polynomial of the optimal system (1), (40), and $D^*(s)$ the characteristic polynomial of the original system (1), (2). If ν_i and τ_i $(i = \overline{1, n})$ are the roots of these polynomials, then

$$D_\varepsilon(s)=\prod_{i=1}^{n}(s-\nu_i),\quad D^*(s)=\prod_{i=1}^{n}(s-\tau_i). \tag{A.8}$$

From the hypothesis of the boundedness of the integral (10) it follows that

$$\operatorname{Re}\nu_i<0\quad (i=\overline{1,n}). \tag{A.9}$$

For definiteness, suppose that

$$\operatorname{Re}\tau_i\leq 0\quad (i=\overline{1,\delta}),\quad \operatorname{Re}\tau_i\geq 0\quad (i=\delta+\overline{1,n};\ 0<\delta<n). \tag{A.10}$$

Then $\delta = n$, which means that the system (1), (2) under consideration is asymptotically stable.

From (A.6) and (A.9) we find that as $\varepsilon \to 0$,

$$\nu_i\to\tau_i\quad (i=\overline{1,\delta}),\quad \nu_i\to-\tau_i\quad (i=\delta+\overline{1,n}). \tag{A.11}$$

If $\delta = n$, then as $\varepsilon \to 0$, $\nu_i \to \tau_i$ $(i = \overline{1, n})$, and consequently $D_\varepsilon(s) \to D^*(s)$ [11]. This means that if the pair (P, b) is completely controllable, then $c(\varepsilon) \to c^*$.

If $\delta < n$, then $D_\varepsilon(s)\to\prod_{i=1}^{\delta}(s-\tau_i)\prod_{i=\delta+1}^{n}(s+\tau_i)\neq D^*(s)$ [11], and therefore $c(\varepsilon) \to c^{**}$, where c^{**} is a unit vector (by virtue of the fact that (P, b) is completely controllable) whose components can be found by comparing the coefficients of equal powers of s on both sides of the equation

$$d(s)-c^{**'}\tilde{\beta}(s)=\prod_{i=1}^{\delta}(s-\tau_i)\prod_{i=\delta+1}^{n}(s+\tau_i). \tag{A.12}$$

Since the equality

$$d(s)-c^{*'}\tilde{\beta}(s)=\prod_{i=\delta+1}^{n}(s-\tau_i) \tag{A.13}$$

is also satisfied for the unit vector c^*, it follows from (A.12) and (A.13) that $c^{**} \neq c^*$.

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