DETERMINISTIC SYSTEMS

METHOD OF FREQUENCY PARAMETERS

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A new method of analytic design of controllers is proposed for plants specified by finitely many values of their frequency responses.

1. Introduction

The method of logarithmic amplitude-frequency characteristics (the LAFC method) is widely used in controller design practice. This grapho-analytic method makes it possible to visualize the connection of the controller parameters with the accuracy and the performance of the system which is being designed [1].

For using the LAFC method, we need the amplitude-frequency characteristic of the plant (and in the case of nonminimum-phase plants also the phase-frequency characteristic PFC), which in general is assigned in the entire frequency interval from zero to infinity.

To experimentally determine the frequency response in the entire frequency interval is difficult. Therefore it is common to find by experiment the values of the AFC and the PFC for n frequencies (n being the order of the highest derivative of the plant), on the basis of which we can determine (identify) the parameters of the plant transfer function, and then construct the AFC and the PFC for all the frequencies; on the basis of the latter we can find the controller parameters.

Therefore it is necessary to elaborate a direct (nonidentification) method of design that uses the results of experiments with n frequencies without identifying the plant parameters and constructing the AFC and the PFC.

Below we describe such a method for continuous and discrete systems. It is based on a frequency interpretation of modal control.

2. Statement of Problem

Let us consider a plant whose disturbed motion is described in the first approximation by the equation

$$\begin{aligned} &y^{(n)}+d_{n-1}y^{(n-1)}+\ldots+d_1\dot{y}+d_0y= \\ &=k_\gamma u^\gamma+\ldots+k_1\dot{u}+k_0u+m_\alpha f^{(\alpha)}+\ldots+m_1\dot{f}+m_0f, \end{aligned} \tag{1}$$

where y(t), u(t), and f(t) denote the measured variable, the control, and an external disturbance, with $y^{(i)}$, $u^{(j)}$, and $f^{(\ell)}$ ($i = \overline{0, n}$; $j = \overline{0, \gamma}$; $\ell = \overline{0, \alpha}$; $\gamma < n$; $\alpha < n$) being the corresponding derivatives.

By effecting a Laplace transform of (1) with zero initial conditions and by setting f(t) = 0 for the time being, we obtain

$$y(s)=w_0(s)u(s), \tag{2}$$

where

$$w_0(s)=k(s)/d(s),\ s=\lambda+j\omega \tag{3}$$

is the transfer function of the plant (1) with respect to the control. The polynomials occurring in (3) are

$$k(s)=\sum_{i=0}^{\gamma}k_is^i,\quad d(s)=\sum_{i=0}^{n}d_is^i\quad (d_n=1). \tag{4}$$

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Let us introduce (for the time being, formally) the frequency parameters

$$\alpha_k = \operatorname{Re} w_0(\lambda + j\omega_k),\ \beta_k = \operatorname{Im} w_0(\lambda + j\omega_k)\ \ (k = \overline{1, n}),\tag{5}$$

where λ, ω_k $(k = \overline{1, n})$ are assigned numbers.

For $\lambda = 0$ they represent the real and the imaginary frequency responses of the plant. There exists a well-known method [2] of experimental determination of these characteristics. Thus, if the plant is asymptotically stable, then by exciting its input by a harmonic signal $u(t) = 1 \sin \omega_k t$, we obtain at the output

$$y(t) = \operatorname{Re} w_0(j\omega_k) \sin \omega_k t + \operatorname{Im} w_0(j\omega_k) \cos \omega_k t.$$

By applying y(t) to a Fourier filter, we obtain the parameters α_k and β_k $(k = \overline{1, n})$. Let us describe an experimental method of determination of these parameters for an unstable plant for which we have a bound $c_0 \geq 0$ for the largest real part of the roots s_i $(i = \overline{1, n})$ of the polynomial d(s):

$$c_0 > \max_{1 \leq i \leq n} \operatorname{Re} s_i.\tag{6}$$

Assertion 1. For f = 0 let us apply to the input of the plant (1) a signal

$$u(t) = e^{\lambda t} \sin \omega_k t,\ \lambda \geq c_0.\tag{7}$$

Then the output signal modulated by the exponential function $e^{-\lambda t}$

$$\tilde{y}(t) = y(t) e^{-\lambda t},\tag{8}$$

will have the property

$$\lim_{t \to \infty} \tilde{y}(t) = \operatorname{Re} w_0(\lambda + j\omega_k) \sin \omega_k t + \operatorname{Im} w_0(\lambda + j\omega_k) \cos \omega_k t.\tag{9}$$

This assertion is proved in the Appendix.

By virtue of this property we obtain with the aid of a Fourier filter the parameters α_k and β_k.

Definition 1. The frequency parameters of the plant (1) are defined by a collection of 2n numbers which are the coefficients of harmonic and steady-state responses (9) to a signal (7).

In going over to the formulation of the problem, we shall assume that the parameters d_i and k_j $(i = \overline{0, n-1};\ j = \overline{0, \gamma})$ of Eq. (1) are unknown. We know that the plant (1) is completely controllable and that the roots of the polynomial k(s) have negative real parts; the number c_0 is assigned, as well as the bounds $m_i^*(i = \overline{0, \alpha})$

$$|m_i| \leq m_i^*\ (i = \overline{0, \alpha})\tag{10}$$

and the frequency parameters α_k and β_k $(k = \overline{1, n})$. As f(t) we shall take typical [1] step or harmonic disturbances that satisfy the condition

$$|f(t)| \leq f^*,\tag{11}$$

where f^* is an assigned number.

Problem 1. For a plant (1) specified by the frequency parameters α_k and β_k $(k = \overline{1, n})$, find parameters of the controller

$$g_{n-1} u^{(n-1)} + \ldots + g_1 \dot{u} + g_0 u = r_{n-1} y^{(n-1)} + \ldots + r_1 \dot{y} + r_0 y,\tag{12}$$

such that the steady-state error $y_{std} = \lim_{t \to \infty} y(t)$ in system (1), (12) under the action of typical disturbances satisfies the condition

$$|y_{std}| \leq y^*_{std},\tag{13}$$

where y_{std}^* is an assigned number.

This problem differs from the problem studied in [3] only by a possible instability of the plant. The latter does not permit the use of the method (proposed in [3]) of optimal control on the basis of frequency parameters, since it is based on an optimality condition in frequency form which is constructed on the assumption of asymptotic stability of the plant. Moreover, the extension of this method to the case of discrete plants and plants with delay encounters insurmountable difficulties caused by the specific form of the condition of optimality of discrete systems [4].

Below we propose a method that solves Problem 1 and that can be extended in a natural manner to the case of discrete plants and plants with delay in the control.

3. Modal Control Based on Frequency Parameters

At first we shall assume that the parameters d_i and k_j ($i = \overline{0, n-1}$; $j = \overline{0, \gamma}$) of the plant (1) are known and that it is required to find parameters of the controller (12) such that the characteristic polynomial of system (1), (12), i.e.,

$$D(s)=d(s)g(s)-k(s)r(s), \tag{14}$$

where $g(s) = g_{n-1}s^{n-1} + \ldots + g_1 s + g_0$, $r(s) = r_{n-1}s^{n-1} + \ldots + r_1 s + r_0$, coincides with the assigned polynomial

$$\delta(s)=s^{2n-1}+\delta_{2n-2}s^{2n-2}+\ldots+\delta_1 s+\delta_0. \tag{15}$$

It is well known that this modal control problem can be solved as follows.

Let us construct the equation

$$d(s)g(s)-k(s)r(s)=\delta(s) \tag{16}$$

and by comparing in it the coefficients of equal powers of s, we obtain a system of linear algebraic equations for the parameters g_i and r_j ($i = \overline{0, n-1}$; $j = \overline{0, n-1}$) of the controller (12):

$$\sum_{i=0}^{n} d_i g_{\alpha-i}-\sum_{l=0}^{\gamma} k_l r_{\alpha-l}=\delta_\alpha \quad (\alpha=\overline{0,2n-1}), \tag{17}$$

where $g_{\alpha-i} \equiv 0$, for $\alpha - i < 0$ and $\alpha - i > n - 1$; $r_{\alpha-\ell} \equiv 0$, for $\alpha - \ell < 0$ and $\alpha - \ell > n - 1$.

A solution of system (17) exists and it is unique by virtue of the fulfillment of the condition [5]

$$\text{degree of } r(s) < \text{degree of } \alpha(s)\nu^{-1}(s). \tag{18}$$

Here $\nu(s)$ is the greatest common divisor of the polynomials d(s) and k(s). The polynomial $\nu(s) = 1$ by virtue of the complete controllability of the plant (1).

In going over to the case of unknown plant parameters, let us assign a polynomial structure

$$\delta(s)=k(s)\psi(s), \tag{19}$$

where

$$\psi(s)=s^{2n-\gamma-1}+\psi_{2n-\gamma-2}s^{2n-\gamma-2}+\ldots+\psi_1 s+\psi_0. \tag{20}$$

Below we shall assume that all the roots of $\psi(s)$ have negative real parts.

Let us substitute (19) into (16) and divide the result by d(s). By setting $s = \lambda + j\omega$, we then obtain a modal frequency equation

$$g(\lambda+j\omega)-r(\lambda+j\omega)w_0(\lambda+j\omega)=\psi(\lambda+j\omega)w_0(\lambda+j\omega). \tag{21}$$

By equating $\omega = \omega_k$ ($k = \overline{1, n}$) in (21), we obtain a system of equations

$$\sum_{i=0}^{n-1} g_i(\lambda+j\omega_k)^i-w_0(\lambda+j\omega_k)\sum_{i=0}^{n-1} r_i(\lambda+j\omega_k)^i=$$

$$=w_0(\lambda+j\omega_k)\sum_{i=0}^{2n-\gamma-1}\psi_i(\lambda+j\omega_k)^i \quad (k=\overline{1,n}). \tag{22}$$

Let us transform this system taking into account that the sought controller parameters are real. Let us write

$$(\lambda+j\omega)^i=\rho_i(\omega)+j\mu_i(\omega) \quad (i=\overline{0,2n-\gamma}), \tag{23}$$

where

$$\rho_i(\omega)=\sum_{\nu=0}^{[i/2]}\delta_{2\nu}{}^i\omega^{2\nu}, \quad \mu_i(\omega)=\sum_{\nu=0}^{[i/2]}\delta_{2\nu+1}^i\omega^{2\nu+1} \quad (i=\overline{0,2n-\gamma}), \tag{24}$$

[i/2] being the integer part of the number i/2, and $\kappa_{2\nu+1} = 0$ for $2\nu + 1 > i$. The coefficients of these polynomials are functions of λ, i.e.,

$$\delta_{2\nu}{}^i=\lambda^{i-2\nu}(-1)^\nu\xi_{2\nu}^{(1)}, \quad \delta_{2\nu+1}^i=\lambda^{i-2\nu-1}(-1)^\nu\xi_{2\nu+1}^{(2)},$$

where $\xi_{2\nu}{}^{(1)}$ and $\xi_{2\nu+1}{}^{(2)}$ ($\nu = 0$, [i/2]) are known numbers. By substituting (23) and (5) into (22), we obtain a system of 2n linear equations for the 2n parameters of the controller:

$$\begin{aligned}&\sum_{i=0}^{n-1}\rho_i(\omega_k)g_i-\sum_{i=0}^{n-1}[\alpha_k\rho_i(\omega_k)-\beta_k\mu_i(\omega_k)]r_i=\\&=\sum_{i=0}^{2n-\gamma-1}[\alpha_k\rho_i(\omega_k)-\beta_k\mu_i(\omega_k)]\psi_i \quad (k=\overline{1,n}),\end{aligned} \tag{25}$$

$$\begin{aligned}&\sum_{i=0}^{n-1}\mu_i(\omega_k)g_i-\sum_{i=0}^{n-1}[\alpha_k\mu_i(\omega_k)+\beta_k\rho_i(\omega_k)]r_i=\\&=\sum_{i=0}^{2n-\gamma-1}[\alpha_k\mu_i(\omega_k)+\beta_k\rho_i(\omega_k)]\psi_i \quad (k=\overline{1,n}).\end{aligned} \tag{26}$$

Assertion 2. If the plant (1) is completely controllable and its characteristic polynomial

$$d(s_k)\neq 0 \text{ for } s_k=\lambda+j\omega_k \ (k=\overline{1,n}), \tag{27}$$

then a solution of system (25)-(26) a) exists, b) is unique, and c) coincides with the solution of system (17).

This assertion is proved in the Appendix.

The equations (25)-(26) constitute a direct method of design of modal control on the basis of frequency parameters. It is true, though, that the modal control problem solved by this method is somewhat specific, i.e., the polynomial (15) is not just any polynomial, but it has the structure (19).

Together with this we can obviously have also an identification method of modal control based on the frequency parameters. It involves identifying the coefficients of Eq. (1) on the basis of frequency parameters and solving the equations (17) of modal control. These operations require the solving of 2n identification equations, as well as 2n equations (17) for the controller parameters. The direct method requires the solving of only 2n equations (25)-(26) which express the sought controller parameters directly in terms of the frequency parameters, circumventing the identification operation.

4. Method of Frequency Parameters

In proceeding to the solution of Problem 1, let us establish a connection between the input variable y(t) and the disturbance in a system (1), (12) whose controller parameters are specified on the basis of (25)-(26). By virtue of (16) and (19) we obtain from (1), (12) the expression

$$y(s)=\frac{m(s)g(s)}{d(s)g(s)-k(s)r(s)}f(s)=\frac{m(s)g(s)}{k(s)\psi(s)}f(s), \tag{28}$$

where $m(s) = m_\alpha s^\alpha + \dots + m_1 s + m_0$ is the disturbance polynomial in (1).

In the general case the polynomial g(s) is not explicitly depending on $\psi(s)$, and therefore it is difficult to find a method of determination of the polynomial $\psi(s)$ such that the accuracy requirements (13) are satisfied.

In this connection let us note that there exists a structure of the polynomial $\psi(s)$ such that g(s) does not depend on $\psi(s)$. Such a structure, denoted by $\tilde{\psi}(s)$, has the form

$$\tilde{\psi}(s)=s^n+\tilde{\psi}_{n-1}s^{n-1}+\dots+\tilde{\psi}_1 s+\tilde{\psi}_0. \tag{29}$$

Indeed, a solution of the equation

$$d(s)g(s)-k(s)r(s)=k(s)\tilde{\psi}(s) \tag{30}$$

is given by the polynomials

$$g(s)=k(s),\ r(s)=d(s)-\tilde{\psi}(s). \tag{31}$$

By virtue of (31) we shall write the constraint (28) in the form

$$y(s)=\frac{m(s)}{\tilde{\psi}(s)}f(s). \tag{32}$$

By using the bounds (10) of the parameters of the polynomial m(s) and the expressions f(s) for step or harmonic external disturbances, it is easy to find parameters of the polynomial (29) such that the accuracy requirement (13) of system (1), (12) is satisfied. Thus, in the case of a step signal we have

$$y_{\text{std}}\leqslant\frac{m_0^*}{\tilde{\psi}_0}f^*. \tag{33}$$

Here let us note that the solution (31) corresponds to an unrealizable controller if the degree γ of the polynomial k(s) in Eq. (1) is smaller than $n - 1$. Instead of (30) let us therefore consider an equation

$$d(s)g(s)-k(s)r(s)=k(s)\varkappa(s)\tilde{\psi}(s), \tag{34}$$

where

$$\varkappa(s)=\varkappa_{n-\gamma-1}s^{n-\gamma-1}+\dots+\varkappa_1 s+\varkappa_0. \tag{35}$$

By assigning the structure of the polynomial

$$g(s)=\rho(s)k(s), \tag{36}$$

where $\rho(s) = \rho_{n-\gamma-1}s^{n-\gamma-1} + \dots + \rho_1 s + \rho_0$, and by substituting it into (34), we obtain

$$d(s)\rho(s)-r(s)=\varkappa(s)\tilde{\psi}(s). \tag{37}$$

Let us write

$$\varkappa(s)=(\tau s+1)^{n-\gamma-1}. \tag{38}$$

Assertion 3. There always exists a sufficiently small number τ such that (37) is satisfied by the polynomials

$$\rho(s)=\varkappa(s)+O^{\varkappa}(s),\ r(s)=d(s)-\tilde{\psi}(s)+O^{r}(s), \tag{39}$$

where $O^\kappa(s)$, $O^r(s)$ are polynomials of degree $n - \gamma - 1$ and $n - 1$ whose parameters can be made as small as desired by selecting a sufficiently small number τ.

This assertion is proved in the Appendix.

For specifying the number τ, let us proceed as follows. Let us write

$$\tilde{\psi}(s)=(s+s^*)^n, \quad (40)$$

where s^* is an assigned positive number, and assume below that

$$\tau<1/s^*. \quad (41)$$

Usually a controller increases the speed of response of a system, and therefore we often have

$$s^*>|s_i| \ (i=\overline{1, n}), \quad (42)$$

where the s_i are roots of the polynomial d(s).

By using the formulas (A.19)-(A.23) of the Appendix, it is easy to show that under the conditions (41)-(42) the parameters of the polynomials $O^{\kappa}(s)$, $O^{r}(s)$ are sufficiently small. Thus, the frequency parameter method which solves Problem 1 involves the following operations.

1. To find the number s^* on the basis of the accuracy requirement (13). Thus, in the case of an external step signal we obtain by virtue of (33) the expression

$$s^*=\left(\frac{m_0^* f^*}{\dot{y}_{std}}\right)^{1/n}. \quad (43)$$

2. To construct the polynomial

$$\psi(s)=(\tau s+1)^{n-\gamma-1}(s+s^*)^n, \ \tau<1/s^*. \quad (44)$$

3. To solve the equations (25)-(26).

5. Control of Discrete Plants

Let us consider a discrete plant with delayed control

$$\begin{aligned} &y(iT)+d_1 y[(i-1)T]+\ldots+d_n y[(i-n)T]= \\ &=k_\gamma u[(i-\kappa-\gamma)T]+\ldots+k_n u[(i-n-\kappa)T]+ \\ &+m_\alpha f[(i-\alpha)T]+\ldots+m_0 f[(i-n)T] \ (i=0, 1, \ldots), \end{aligned} \quad (45)$$

where T is the discreteness interval, and κT the control delay.

By using the z-transform [6], we obtain for zero initial conditions the transfer function of the plant (45) with respect to the control:

$$w_0(z^{-1})=k(z^{-1})/d(z^{-1}), \ z=e^{(\lambda+j\omega)T}, \quad (46)$$

where

$$k(z^{-1})=z^{-(\kappa+\gamma)}\sum_{i=\gamma}^{n} k_i z^{-(i-\gamma)}, \quad d(z^{-1})=1+\sum_{i=1}^{n} d_i z^{-i}.$$

The plant frequency parameters

$$\alpha_k=\mathrm{Re}\, w_0(e^{-(\lambda+j\omega_k)T}), \ \beta_k=\mathrm{Im}\, w_0(e^{-(\lambda+j\omega_k)T}) \ (k=\overline{1, N}) \quad (47)$$

can be determined experimentally by applying to the input of the plant (45) a signal

$$u(iT)=e^{\lambda_i T}\sin\omega_i T \ (i=0, 1, \ldots), \quad (48)$$

and by letting the output signal modulated by an exponential function $e^{-\lambda iT}$ pass through a discrete Fourier filter. The number λ can be determined from the inequality

$$e^{\lambda T}\geqslant z_0>\max_{1\leqslant i\leqslant n}|z_i|. \quad (49)$$

Here z_0 is a bound of the absolute values of the roots z_i of the polynomial $d(z^{-1})$.

Suppose that it is required to find a controller

$$g(z^{-1})u=r(z^{-1})y, \tag{50}$$

where $g(z^{-1})=\sum_{i=0}^{\eta} g_i z^{-i}$, $r(z^{-1})=\sum_{i=0}^{\nu} r_i z^{-i}$, that solves Problem 1 for the plant (45).

Similarly to (16) and (19) we shall then write the modal equation and the objective polynomial

$$d(z^{-1})g(z^{-1})-k(z^{-1})r(z^{-1})=k(z^{-1})\psi(z^{-1}), \tag{51}$$

where

$$\psi(z^{-1})=1+\sum_{i=1}^{\xi}\psi_i z^{-i}, \quad \xi=2n-\varkappa. \tag{52}$$

As before, we shall assume that the plant is completely controllable and that the roots z_i^{-1} ($i = \overline{1, \gamma+\kappa}$) of the polynomial $k(z^{-1})$ do not exceed unity in absolute value.

By virtue of (46), we obtain from (51)-(52) the modal frequency equation

$$g(e^{-(\lambda+j\omega)T})-w_0(e^{-(\lambda+j\omega)T})=w_0(e^{-(\lambda+j\omega)T})\psi(e^{-(\lambda+j\omega)T}). \tag{53}$$

By setting $\omega = \omega_k$ ($k = \overline{1, n}$) in (53), we obtain a system of linear equations for the parameters of the controller (50). For satisfying the accuracy requirement (13), we shall take $\xi = n$ in (52). Then (51) will have a solution

$$g(z^{-1})=k(z^{-1}), \; r(z^{-1})=d(z^{-1})-\psi(z^{-1}), \tag{54}$$

such that

$$y(z)=\frac{m(z^{-1})}{\psi(z^{-1})}f(z). \tag{55}$$

Hence we obtain the values of the parameters of the polynomial $\psi(z^{-1})$.

6. Multidimensional Plants

Let us consider a multidimensional plant described by Eq. (1), in which y(t) and u(t) are m-dimensional vectors, and d_i and k_j ($i = \overline{0, n-1}$; $j = \overline{0, \gamma}$) are ($m \times m$)-dimensional number matrices, whereas the m_i ($i = \overline{0, \alpha}$) are m-dimensional vectors.

The plant transfer matrix is

$$w_0(s)=d^{-1}(s)k(s), \; s=\lambda+j\omega, \tag{56}$$

where d(s) and k(s) are matrix polynomials of degree n and γ, respectively, $d(s)d^{-1}(s) = E_m$ (E_m being a unit matrix of dimension $m \times m$).

The frequency parameters form the matrices α_k and β_k of frequency parameters. The elements $\alpha_k^{i\nu}$ and $\beta_k^{i\nu}$ ($i, \nu = \overline{1, m}$; $k = \overline{1, m}$) of these matrices can be obtained experimentally as follows: By applying to the ν-th input of the plant a signal $u^{\nu}(t) = e^{\lambda t}\sin\omega_k t$ and by letting the i-th plant output signal modulated by an exponential function $e^{\lambda t}$ pass through a Fourier filter, we obtain the numbers $\alpha_k^{i\nu}$ and $\beta_k^{i\nu}$. The exponent λ is a bound of the largest real part of the roots of the polynomial $\det d(s)$.

Suppose that it is required to find matrices of a controller of type (12) such that the accuracy requirement (13) holds with respect to the multidimensional output y.

Let us construct an equation

$$d(s)-k(s)g^{-1}(s)r(s)=k(s)g^{-1}(s)\psi(s), \tag{57}$$

which after multiplication by g(s) coincides with (16) in the one-dimensional case (m = 1).

It is easy to show that if (57) holds, then the vectors y and m(s)f are connected by the following relation similar to (28):

$$y(s)=\psi^{-1}(s)g(s)k^{-1}(s)m(s)f(s). \tag{58}$$

By multiplying (57) at first from the left by $g(s)k^{-1}(s)$, and then from the right by $d^{-1}(s)k(s)$, we obtain a matrix frequency modal equation of type (21). By setting $\omega = \omega_k$ $(k = \overline{1, n})$ in it, we obtain a system of 2nm linear algebraic equations for 2nm controller matrix elements.

For meeting the accuracy requirements of the system which is being designed, we shall assume that the degree of the matrix polynomial $\psi(s)$ is equal to n. Then (57) will have an obvious solution

$$g(s)=k(s),\ r(s)=d(s)-\psi(s), \tag{59}$$

such that (58) takes the form

$$y(s)=\psi^{-1}(s)m(s)f(s).$$

With the aid of this relation we can determine the matrices of the polynomial $\psi(s)$ on the basis of the accuracy requirements.

7. Example

Let us consider a completely controllable minimum-phase plant

$$\ddot{y}+d_1\dot{y}+d_0y=k_1\dot{u}+k_0u+f \tag{60}$$

with unknown parameters d_1, d_0, k_1, and k_0. We know the frequency parameters of the plant (60)

$$\alpha_1=0.85,\ \beta_1=-1.4,\ \alpha_2=0.22,\ \beta_2=-0.88, \tag{61}$$

obtained experimentally for

$$\lambda=6,\ \omega_1=3\cdot 1/\text{sec},\ \omega_2=6\cdot 1/\text{sec}. \tag{62}$$

The external disturbance f(t) is a step function

$$f(t)=\begin{cases}0 & \text{for } t<t_0,\\ \bar{f} & \text{for } t\geqslant t_0,\end{cases}\qquad |\bar{f}|\leqslant f^*, \tag{63}$$

where $f^* = 10$.

It is required to find a controller

$$g_1\dot{u}+g_0u=r_1\dot{y}+r_0y, \tag{64}$$

such that the steady-state error in the system (60), (64) is

$$|y_{\text{std}}|\leqslant 0.204. \tag{65}$$

In accordance with (43) we obtain $s^* = 7$. Since in the case under consideration we have $n = 2$ and $\gamma = 1$, let us find a polynomial (44) of the form $\psi(s) = s^2 + 14s + 49$, and hence,

$$\psi_1=14,\ \psi_0=49. \tag{66}$$

By virtue of the obvious relations

$$\rho_0(\omega)=1,\ \mu_0(\omega)=0,\ \rho_1(\omega)=\lambda,\ \mu_1(\omega)=\omega,$$
$$\rho_2(\omega)=\lambda^2-\omega^2,\ \mu_2(\omega)=2\lambda\omega$$

we can write the equations (25)-(26) in the form

$$\begin{aligned}&g_0+\lambda g_1-\alpha_k r_0-(\alpha_k\lambda-\beta_k\omega_k)r_1=\\&=\alpha_k\psi_0+(\alpha_k\lambda-\beta_k\omega_k)\psi_1+[\alpha_k(\lambda^2-\omega_k^2)-2\beta_k\lambda\omega_k],\end{aligned} \tag{67}$$

$$\lambda g_1-\beta_k r_0-(\alpha_k\omega_k+\beta_k\lambda)r_1=\beta_k\psi_0+(\alpha_k\omega_k+\beta_k\lambda)\psi_1+ \\ +[\alpha_k 2\lambda\omega_k+\beta_k(\lambda^2-\omega_k^2)] \quad (k=\overline{1,2}). \tag{68}$$

By substituting the parameters (61), (62), and (66) into (67)-(68) and by solving this system of four linear algebraic equations, we obtain the coefficients of (64):

$$g_1=5,\ g_0=30,\ r_0=-65,\ r_1=-14. \tag{69}$$

Appendix

Proof of Assertion 1. Let us reduce (1) to Cauchy form:

$$\dot{x}=Px+bu, \tag{A.1}$$

$$y=l'x, \tag{A.2}$$

where x(t) is an n-dimensional vector, P is a number matrix of dimension n × n, and b and ℓ are n-dimensional vectors.

Let us find the response of (A.1) to a signal (7):

$$u(t)=e^{\lambda t}\sin\omega t=\frac{e^{(\lambda+j\omega)t}-e^{(\lambda-j\omega)t}}{2j}=\frac{e^{st}-e^{\bar{s}t}}{2j}.$$

For this purpose we shall consider the equation

$$\dot{x}^+=Px^++be^{st}. \tag{A.3}$$

Its solution is

$$x^+(t)=\int_{t_0}^{t}e^{P(t-\tau)}be^{s\tau}\,d\tau=e^{Pt}\int_{t_0}^{t}e^{(Es-P)\tau}b\,d\tau=\beta(s)e^{st}-e^{Pt}\beta(s), \tag{A.4}$$

where $\beta(s) = (Es - P)^{-1}b$, $t_0 = 0$.

By replacing s by $\bar{s}$ in (A.4), we obtain a solution of the equation

$$\dot{x}^-=Px^-+be^{\bar{s}t}.$$

By virtue of (A.2) we then write

$$y=l'x=l'\frac{x^+-x^-}{2j}=\frac{[l'\beta(s)e^{j\omega t}-l'\beta(\bar{s})e^{-j\omega t}]e^{\lambda t}}{2j}-\frac{l'e^{Pt}[\beta(s)-\beta(\bar{s})]}{2j}.$$

Since $\ell'\beta(s) = w_0(s)$, we obtain

$$y(t)=[\mathrm{Re}\,w_0(s)\sin\omega t+\mathrm{Im}\,w_0(s)\cos\omega t]e^{\lambda t}-l'e^{Pt}\,\mathrm{Im}\,\beta(s). \tag{A.5}$$

By constructing $\tilde{y}(t) = y(t)e^{-\lambda t}$ and by noting that the definition of the number λ yields

$$\lim_{t\to\infty}l'e^{(-E\lambda+P)t}=0, \tag{A.6}$$

we obtain (9).

Proof of Assertion 2. For this proof let us show that the equations (25) and (26) coincide under the condition (27) with the equations (17) whose solution exists and is unique for a completely controllable plant.

By multiplying (26) by j and by adding the product to (25), we obtain (22). After multiplying the latter by $d(s_k)$ $(k = \overline{1, n})$, we obtain a system

$$d(s_k)g(s_k)-k(s_k)r(s_k)=\delta(s_k) \quad (k=\overline{1,n}), \tag{A.7}$$

which can be easily written as

$$\sum_{\alpha=0}^{2n-1}(\lambda+j\omega_k)^\alpha \varkappa_\alpha=0, \qquad (A.8)$$

where

$$\varkappa_\alpha=\sum_{i=0}^{n} d_i g_{\alpha-i}-\sum_{l=0}^{7} k_l r_{\alpha-l}-\delta_\alpha \quad (\alpha=\overline{0, 2n-1}). \qquad (A.9)$$

If the solution of (A.8) is

$$\varkappa_\alpha=0 \quad (\alpha=\overline{0, 2n-1}), \qquad (A.10)$$

then (17) follows from (A.9).

In proceeding to the proof of (A.10), let us at first consider the case $\lambda = 0$.

Since κ_α ($\alpha = \overline{0,\ 2n-1}$) is real, it follows that in this case (A.8) can be expressed by

$$\sum_{i=0}^{n-1}\omega^{2i}(-1)^i\varkappa_{2i}=0, \quad \sum_{i=0}^{n-1}\omega^{2i+1}(-1)^i\varkappa_{2i+1}=0 \qquad (A.11)$$

or in matrix form

$$M\varkappa=0, \qquad (A.11')$$

where

$$M=\begin{bmatrix} 1 & 0 & -\omega_1^2 & 0 & \omega_1^4\dots & 0 \\ \vdots & & & & \vdots & \vdots \\ 1 & 0 & -\omega_n^2 & 0 & \omega_n^4\dots & 0 \\ 0 & \omega_1 & 0 & -\omega_1^3 & 0\dots & 0 \quad \omega_1^{2n-1}(-1)^{n-1} \\ \vdots & & & & \vdots & \vdots \\ 0 & \omega_n & 0 & -\omega_n^3 & 0\dots & 0 \quad \omega_n^{2n-1}(-1)^{n-1} \end{bmatrix}, \quad \varkappa=\begin{bmatrix} \varkappa_0 \\ \vdots \\ \varkappa_{n-1} \\ \varkappa_n \\ \vdots \\ \varkappa_{2n-1} \end{bmatrix}. \qquad (A.12)$$

It follows from the structure of (A.12) that the determinant of M can be easily represented (by a permutation of columns) in the form

$$\det M=\det\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}=\det w_1\cdot\det w_2\neq 0, \qquad (A.13)$$

since w_1 and w_2 are ($n \times n$)-dimensional matrices whose determinants are of Vandermonde type [7], and

$$\det w_1\neq 0, \quad \det w_2\neq 0, \quad \text{for} \quad \omega_\alpha\neq\omega_\beta \ (\alpha\neq\beta). \qquad (A.14)$$

Thus we have proved (A.10) for $\lambda = 0$. For $\lambda \neq 0$ the equations (A.8) take by virtue of (24) the form

$$\sum_{\alpha=0}^{2n-1}\left(\sum_{i=0}^{[\alpha/2]}\delta_{2i}{}^\alpha\omega_k{}^{2i}\right)\varkappa_\alpha=0, \quad \sum_{\alpha=0}^{2n-1}\left(\sum_{i=0}^{[\alpha/2]}\delta_{2i+1}^{\alpha}\omega_k^{2i+1}\right)\varkappa_\alpha=0 \quad (k=\overline{1, n})$$

or the matrix form

$$\bar{M}\varkappa=0, \qquad (A.15)$$

where

$$\bar{M}=\begin{bmatrix} 1, & \lambda, & \lambda^2-\omega_1^2, & \delta_0^3+\delta_2^3\omega_1^2+\ldots, & \sum_{i=0}^{n-2}\delta_{2i}^{2n-1}\omega_1^{2i}+\omega_1^{2n-2}(-1)^{n-1} \\ \vdots & & & & \\ 1, & \lambda, & \lambda^2-\omega_n^2, & \delta_0^3+\delta_2^3\omega_n^2+\ldots, & \sum_{i=0}^{n-2}\delta_{2i}^{2n-1}\omega_n^{2i}+\omega_n^{2n-2}(-1)^{n-1} \\ 0, & \omega_1, & \lambda\omega_1, & \delta_1^3\omega_1+\omega_1^2\ldots, & \sum_{i=0}^{n-2}\delta_{2i+1}^{2n-1}\omega_n^{2i+1}+\omega_1^{2n-1}(-1)^{n-1} \\ \vdots & & & & \\ 0, & \omega_n, & \lambda\omega_n, & \delta_1^3\omega_n+\omega_n^3\ldots, & \sum_{i=0}^{n-2}\delta_{2i+1}^{2n-1}\omega_n^{2i+1}+\omega_n^{2n}(-1)^{n-1} \end{bmatrix}. \tag{A.16}$$

Let us show that

$$\det \bar{M}=\det M. \tag{A.17}$$

For this purpose let us effect a transformation of the matrix M that does not change its determinant. We multiply the second column of $\bar{M}$ by λ and subtract it from the third column; then we multiply the first column of $\bar{M}$ by λ and subtract it from the second column. Thus, the first three columns of $\bar{M}$ will coincide with the corresponding columns of the matrix M. Such a transformed matrix $\bar{M}$ will be denoted by $\bar{M}^{(1)}$. Let us multiply the first column of $\bar{M}^{(1)}$ by $\delta_0{}^3$, the second by $\delta_1{}^3$, and the third by $\delta_2{}^3$; by subtracting them from the fourth column, we obtain the fourth column of the matrix M. By continuing this process, we obtain (A.17) and thus also (A.10); hence we have proved Assertion 2.

<u>Proof of Assertion 3.</u> For simplicity we shall confine ourselves to the case $n-\gamma-1=2$. Then (37) will take the form

$$d(s)(\rho_2 s^2+\rho_1 s+\rho_0)-r(s)=(\tau^2 s^2+2\tau s+1)\tilde{\psi}(s). \tag{A.18}$$

By comparing the coefficients of equal powers of s, we obtain equations for the parameters of the polynomials $\rho(s)$ and $r(s)$:

$$\begin{aligned}
&s^{n+2}: \rho_2-\tau^2=0=O_2^{\varkappa},\\
&s^{n+1}: \rho_1-2\tau=\tilde{\psi}_{n-1}\tau^2-d_{n-1}\rho_2=O_1^{\varkappa},\\
&s^{n}: \rho_0-1=\tilde{\psi}_{n-1}2\tau+\tilde{\psi}_{n-2}\tau^2-d_{n-1}\rho_1-d_{n-2}\rho_2=O_0^{\varkappa},\\
&s^{n-1}: -r_{n-1}+d_{n-1}\rho_0-\tilde{\psi}_{n-1}=d_{n-2}\rho_1-d_{n-3}\rho_2+\tilde{\psi}_{n-2}2\tau+\tilde{\psi}_{n-3}\tau^2=O_{n-1}^{r},\\
&\vdots\\
&s^{2}: -r_2+d_2\rho_0-\tilde{\psi}_2=d_1\rho_1-d_0\rho_2+\tilde{\psi}_1 2\tau+\tilde{\psi}_0\tau^2=O_2^{r},\\
&s^{1}: -s_1+d_1\rho_0-\tilde{\psi}_1=d_0\rho_1+\tilde{\psi}_0 2\tau=O_1^{r},\\
&s^{0}: -r_0+d_0\rho_0-\tilde{\psi}_0=0=O_0^{r}.
\end{aligned}$$

From the equations for the coefficients of s^{n+2}, s^{n+1}, and s^n it follows that

$$\rho_2=\tau^2,\quad \rho_1=2\tau+O_1{}^{\varkappa}(\tau^2),\quad \rho_0=1+O_0{}^{\varkappa}(\tau), \tag{A.19}$$

where $\lim_{\tau\to\infty} O_1{}^{\kappa}(\tau^2)/\tau=0$, $\lim_{\tau\to 0} O_0{}^{\kappa}(\tau)=0$.

From the equations for the coefficients of s^{n-1}, ..., s^2, s^1, s^0 we similarly obtain

$$r_{n-1}=\tilde{\psi}_{n-1}-d_{n-1}+O_{n-1}^{r}, \tag{A.20}$$

where

$$\begin{aligned}
&O_{n-1}^{r}=\tilde{\psi}_{n-3}\tau^2+\tilde{\psi}_{n-2}2\tau-d_{n-3}\tau^2-d_{n-2}[2\tau+O_1^{\varkappa}(\tau^2)]-d_{n-1}O_0^{\varkappa}(\tau),\\
&\vdots\\
&r_2=\tilde{\psi}_2-d_2+O_2^{r},
\end{aligned} \tag{A.21}$$

$$O_2^r=\widetilde{\psi}_0\tau^2+\widetilde{\psi}_1 2\tau-d_0\tau^2-d_1[2\tau+O_1^\varkappa(\tau^2)]-d_2O_0^\varkappa(\tau), \quad r_1=\widetilde{\psi}_1-d_1+O_1^r, \qquad (A.22)$$

$$O_1^r=\widetilde{\psi}_0 2\tau-d_0[2\tau+O_1^\varkappa(\tau^2)]-d_1O_0^\varkappa(\tau), \quad r_0=\widetilde{\psi}_0-d_0+O_0^r, \quad O_0^r=d_0O_0^\varkappa(\tau). \qquad (A.23)$$

These relations yield

$$\lim_{\tau\to 0} O_i^r(\tau)=0 \quad (i=\overline{0, n-1}),$$

and thus we have proved the assertion.

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LOCAL DATA PROCESSING ALGORITHMS IN LINEAR SYSTEMS

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The state of a linear system is determined on the basis of measurements of the output in a given finite interval. By analyzing the possible use of linear observers for this purpose, it was found that the observer gains depend on the time, and they have pole-type singularities at the initial or final point of the interval. Therefore two types of observers are considered, and the properties of observers with strong data processing in a neighborhood of the initial instant of measurement are studied.

1. Statement of Problem

Let us consider a linear system

$$\dot{x}=A(t)x, \; y=c(t)x, \; x(t_0)=x_0 \qquad (1)$$

and it is required to determine a vector $x(t_f)$ on the basis of a function $y = y(t)$ measured at $t \in I = [t_0, t_f]$. Here $x \in R^n$, $y \in R^1$, $A(t)$ is an $(n \times n)$-matrix, and $c(t)$ a $(1 \times n)$-matrix. One of the possible solutions of this problem is well known (see, for example, [1]), and it can be written as follows:

$$x(t_f)=M^{-1}(t_0, t_f)\int_{t_0}^{t_f} \Phi^T(\theta, t_f)c^T(\theta)y(\theta)d\theta. \qquad (2)$$

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