TECHNICAL NOTES

FREQUENCY CHARACTERISTICS OF OPTIMAL CONTROL SYSTEMS

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UDC 62-

Single control systems that are optimal in the sense of a definite-positive quadratic functional are considered. It is shown, in particular, that the stability phase margin at cutoff frequency is not less than 60° and the oscillatability index does not exceed two. The relation between the coefficients of the optimized functional, on one hand, and the crossover gain of the optimal system in the open state, on the other hand, is also determined.

1. Definitions and Notation

We consider a control system in which the asymptotically stable perturbed motion is described by the equat[ions]

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}x_j + b_i u \quad (i = 1, \ldots, n), \tag{1.1}$$

$$u = \sum_{i=1}^{n} c_i x_i, \tag{1.2}$$

where x_i $(i = 1, \ldots, n)$ are the phase coordinates of the controlled item, u are control coordinates, and p_{ij}, b_i, and c_i are numbers.

We assume that the item (1.1) is fully controlled, and the fully satisfied control law (1.2) is such that in the solutions of (1.1) and (1.2) the functional

$$I = \int_0^\infty \left(\sum_{i,j=1}^{n} q_{ij}x_i x_j + u^2 \right) dt, \quad \sum_{i,j=1}^{n} q_{ij}x_i x_j > 0 \quad \text{for all } x \tag{1.3}$$

is minimized for any given initial conditions.

The latter means [1] that the coefficients c_i $(i = 1, \ldots, n)$ of the control equation (1.2) satisfy the following relations:

$$\sum_{\alpha=1}^{n} A_{i\alpha} b_\alpha = c_i \quad (i = 1, \ldots, n), \tag{1.4}$$

$$\sum_{\alpha=1}^{n} (A_{\alpha i} p_{\alpha j} + A_{\alpha j} p_{\alpha i}) - c_i c_j + q_{ij} = 0 \quad (i, j = 1, \ldots, n), \quad A_{ij} = A_{ji}, \tag{1.5}$$

where $\| A_{ij} \|_1^n$ is a definite-positive matrix.

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Equations (1.4) and (1.5) are optimal algebraic conditions. These equations are obtained as a result of synthesis that is optimal in the sense of (1.3) of the control (1.2) on the basis of the Lyapunov-Bellman method [2].

The algebraic optimal criterion is obtained in a form somewhat different from (1.4) and (1.5) in [3].

In going over to the Laplace transforms, we write the transfer function of the open system (1.1) and (1.2) in the form

$$w(s) = K(s)\,/\,D_p(s), \tag{1.6}$$

$$D_p(s) = \begin{vmatrix} s-p_{11} & -p_{12} & \cdots & -p_{1n} & 0 \\ -p_{21} & s-p_{22} & \cdots & -p_{2n} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_{n1} & -p_{n2} & \cdots & s-p_{nn} & 0 \\ c_1 & c_2 & \cdots & c_n & 1 \end{vmatrix}, \tag{1.7}$$

$$K(s) = \begin{vmatrix} s-p_{11} & -p_{12} & \cdots & -p_{1n} & b_1 \\ -p_{21} & s-p_{22} & \cdots & -p_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_{n1} & -p_{n2} & \cdots & s-p_{nn} & b_n \\ c_1 & c_2 & \cdots & c_n & 0 \end{vmatrix}. \tag{1.8}$$

Physically $w(s)$ is the ratio $u(s)/\mu(s)$, where $\mu(s)$ is a certain force applied to the input of the item of the open state control system.

Expanding the determinant (1.8) in elements of the last row, and then in elements of the last column, we write (1.6) in the form

$$w(s) = \sum_{i=1}^{n} c_i \left[\sum_{j=1}^{n} b_j \frac{r_{ij}(s)}{D_p(s)} \right], \tag{1.9}$$

where $r_{ij}(s)$ $(i, j, \ldots, n)$ is an algebraic addition to the element of the i-th column and j-th row of determinant of (1.8).

We introduce a certain function $H_k(s)$:

$$H_k(s) = \sum_{i=1}^{n} h_{ik} \left[\sum_{j=1}^{n} b_j \frac{r_{ij}(s)}{D_p(s)} \right], \tag{1.10}$$

where h_{ik} $(i, k = 1, \ldots, n)$ are coefficients of linear forms that satisfy the condition

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} h_{ki} x_i \right) \left(\sum_{j=1}^{n} h_{kj} x_j \right) = \sum_{i,j=1}^{n} q_{ij} x_i x_j.$$

We use the notation

$$w(j\omega)\,w(-j\omega) = A^2(\omega), \quad \arg w(j\omega) = \varphi(\omega), \quad \sum_{k=1}^{n} H_k(-j\omega) H_k(j\omega) = B^2(\omega). \tag{1.11}$$

Here $A(\omega)$ and $\varphi(\omega)$ are the amplitude and the phase frequency characteristics of the open state system.

In [1] the optimal control conditions (1.2) were obtained in frequency form. The necessary and sufficient condition for the optimal condition (1.2) in the sense of functional (1.3) is the satisfaction of the equation

$$[1 + w(-j\omega)][1 + w(j\omega)] = 1 + \sum_{k=1}^{n} H_k(-j\omega) H_k(j\omega) \tag{1.12}$$

for all real frequencies ω.

2. Frequency Characteristics of Optimal Systems Independent of the Concrete Set of Parameters $[q_{ij}]$ of the Optimization Functional

We write (1.12) with allowance for the notation of (1.11) in the form of the following equations equivalent to (1.12):

$$w(-j\omega)+w(j\omega)+u(-j\omega)w(j\omega)=B^2(\omega), \tag{2.1}$$

$$2A(\omega)\cos\varphi(\omega)+A^2(\omega)=B^2(\omega). \tag{2.1'}$$

On the basis of (2.1) and (2.1') we obtain the following characteristics.

1. The phase margin of the optimal system (1.1) and (1.2) is not less than 60° at the cutoff frequency (ω_c).

The phase margin (φ^*) at cutoff frequency, i.e., at the frequency for which the equality $A(\omega_c) = 1$ is satisfied, is determined by the relation $\varphi^* = \pi + \varphi(\omega_c)$.

The notion of stability margin in the modulus and phase is introduced for those systems in which the real parts of the roots of the equation $D_p(s) = 0$ are nonpositive; in addition, it is assumed that ω_c is unique [4, 5].

Taking into account the fact that $B^2(\omega) \geq 0$, we obtain from (2.1')

$$2A(\omega)\cos\varphi(\omega)+A^2(\omega)\geqslant 0, \tag{2.2}$$

from which we obtain at $\omega = \omega_c$

$$\cos\varphi(\omega_c)\geqslant -0.5. \tag{2.3}$$

The latter means that $\varphi(\omega_c)$ satisfies one of the following relations:

a) $2\pi n\leqslant\varphi(\omega_c)\leqslant(2n+1)\pi-60°$,

b) $-(2n+1)\pi+60°\leqslant\varphi(\omega_c)\leqslant-2\pi n$,

c) $(2n+1)\pi+60°\leqslant\varphi(\omega_c)\leqslant 2(n+1)\pi$,

d) $-2(n+1)\pi\leqslant\varphi(\omega_c)\leqslant-(2n+1)\pi-60°$. (2.4)

From the stability condition ($\varphi(\omega_c) > -\pi$) for systems that are stable and neutral in the open state, it follows that relation (2.4d) must be dropped, while (2.4b) takes the form:

$$-\pi+60°\leqslant\varphi(\omega_c)\leqslant 0. \tag{2.5}$$

On the basis of (2.2), (2.4a, b), and (2.5) we conclude that

$$\varphi^*\geqslant 60°.$$

2. The modulus margin of system (1.1) and (1.2) attains an infinitely high value if the automatic phase response (APR) of this system is of the first order, and is not less than two if the APR is of the second order.

The modulus margin (L) is determined [5] for static and astatic systems by the relation

$$L=\min\left\{A(\omega_1);\ \frac{1}{A(\omega_2)}\right\},$$

where ω_1 and ω_2 obey the equalities

$$\operatorname{Im} w(j\omega_1)=0,\quad \operatorname{Re} w(j\omega_1)<-1,\quad \operatorname{Im} w(j\omega_2)=0,\quad \operatorname{Re} w(j\omega_2)>-1$$

(for APR of the first order, the frequency ω_1 does not exist).

We obtain from (1.12) for all $\omega = \bar\omega$ that satisfy the equation $\operatorname{Im} w(j\bar\omega) = 0$

$$2\operatorname{Re} w(j\bar\omega)+[\operatorname{Re} w(j\bar\omega)]^2=B^2(\omega),$$

or

$$\operatorname{Re} w(j\bar\omega)=-1\pm\sqrt{1+B^2(\bar\omega)},$$

from which it follows directly that APR$-w(j\omega)$ does not intersect the interval (−2.0) of the real axis of the complex plane w, and this in turn proves the statement above.

3. The oscillatability index of the system (1.1) and (1.2) does not exceed 2.

The value of the oscillatability index M is determined [5] by the relation

$$M = \max_{0 \leq \omega \leq \infty} \frac{\operatorname{mod} w(j\omega)}{\operatorname{mod}[1 + w(j\omega)]}. \tag{2.6}$$

On the basis of (2.1) we can write

$$\operatorname{mod} w(j\omega) = A(\omega) = -\cos\varphi(\omega) + \sqrt{\cos^2\varphi(\omega) + B^2(\omega)} \tag{2.7}$$

(the minus sign in front of the radical is dropped because $A(\omega) \geq 0$).

It follows from (1.12) that

$$\operatorname{mod}[1 + w(j\omega)] = \sqrt{1 + B^2(\omega)}. \tag{2.8}$$

Thus

$$\begin{aligned} M &= \max_{0 \leq \omega \leq \infty} \frac{\operatorname{mod} w(j\omega)}{\operatorname{mod}[1 + w(j\omega)]} = \max_{0 \leq \omega \leq \infty} \frac{-\cos\varphi(\omega) + \sqrt{\cos^2\varphi(\omega) + B^2(\omega)}}{\sqrt{1 + B^2(\omega)}} \\ &\leq \max_{\substack{0 \leq \cos\varphi(\omega) \leq 1 \\ 0 \leq B^2(\omega) \leq \infty}} \frac{-\cos\varphi(\omega) + \sqrt{\cos^2\varphi(\omega) + B^2(\omega)}}{\sqrt{1 + B^2(\omega)}} = \max_{0 \leq B^2(\omega) \leq \infty} \frac{1 + \sqrt{1 + B^2(\omega)}}{\sqrt{1 + B^2(\omega)}} = 2. \end{aligned} \tag{2.9}$$

3. Relation Between Coefficients of the Optimized Functional, the Gain, and the Cutoff Frequency of the Open State System

Among the natural requirements of control systems, there are the requirements concerning the values of the gain and cutoff frequency of the open state system. These turn out to be additional limitations on the choice of the form of (1.2), which, in addition, must together with (1.1) yield a minimum of functional (1.3). The coefficients of functional (1.3) are usually known to an accuracy up to certain coefficients, whose arbitrary choice may be used to satisfy in the system (1.1), (1.2) the given additional limitations. In connection with this, it is interesting to establish in clear form the relations between the coefficients of functional (1.3), on one hand, and the gain and cutoff frequency of the open system, on the other hand.

We introduce a certain frequency $\omega_c{}^*$ determined from the equality

$$B^2(\omega_c{}^*) = 1, \tag{3.1}$$

and we show that

$$\operatorname{mod} \lg \frac{\omega_c{}^*}{\omega_c} \leq 0.4. \tag{3.2}$$

Indeed, it follows from (2.7) that at $\omega = \omega_c{}^*$

$$A(\omega_c^*) = -\cos\varphi(\omega_c^*) + \sqrt{\cos^2\varphi(\omega_c^*) + 1}. \tag{3.3}$$

After taking into account the inequality $0 \leq \cos\varphi(\omega) \leq 1$ it follows from (3.3) that

$$0.4 \leq A(\omega_c^*) \leq 2.4, \quad \text{or} \quad \operatorname{mod} 20 \lg A(\omega_c^*) \leq 8 \text{ dB}. \tag{3.4}$$

Assuming the slope of the linear automatic frequency response in the vicinity of ω_c to be not less than 20 dB/decade, we obtain (3.2) on the basis of (3.4).

Thus, if the coefficients q_{ij} are chosen such that the equality

$$B^2(\omega_c^*) = 1 \tag{3.5}$$

is satisfied, then the cutoff frequency of the open system (1.1) and (1.2) differs from $\omega_c{}^*$ by no more than 0.4 decades (by a factor of 2.5).

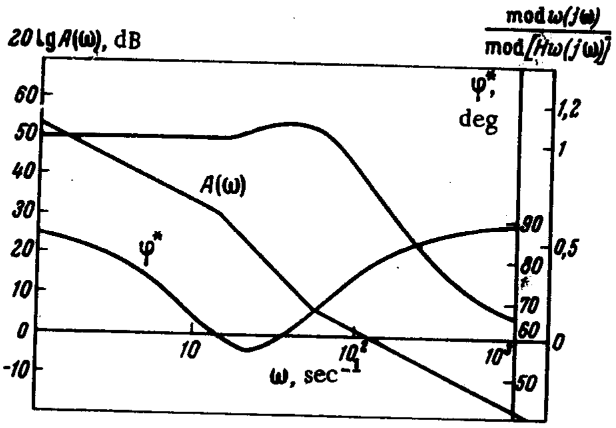

Fig. 1

The gain of the static open state systems, K_p (the transfer coefficient), is determined by the equation

$$K_p = A(0) = w(0). \tag{3.6}$$

At $\omega = 0$ the identity (1.12) has the form

$$[1 + w(0)][1 + w(0)] = 1 + \sum_{k=1}^{n} H_k^2(0). \tag{3.7}$$

Usually $K_p \gg 1$, so that it follows from (3.7) that

$$K_p^2 \approx B^*(0). \tag{3.8}$$

In systems with astatism of r-th order, K_p is determined [5] from the value of the free term of the numerator of the transfer function of the open system of the form

$$w(s) = \frac{K_p(1 + a_1 s + \ldots + a_t s^{(t)})}{s^{(r)}(1 + d_1 s + \ldots + d_m s^{(m)})} \quad (t \leqslant m + r;\ r = 1, \ldots, \nu). \tag{3.9}$$

Introducing the notation

$$\sum_{i=1}^{n} h_{ki}\left[\sum_{i=1}^{n} b_i r_{ij}(s)\right] = L_k(s), \tag{3.10}$$

we write down (1.12), with account taken of (1.6), (1.8), and (3.10), in the form

$$\left[1 + \frac{K(s)}{D_p(s)}\right]\left[1 + \frac{K(-s)}{D_p(-s)}\right] = 1 + \frac{\sum_{k=1}^{n} L_k(s) L_k(-s)}{D_p(s) D_p(-s)}. \tag{3.11}$$

From this it follows that

$$[D_p(s) + K(s)][D_p(-s) + K(-s)] = D_p(s) D_p(-s) + \sum_{k=1}^{n} L_k(s) L_k(-s). \tag{3.12}$$

We note that Eq. (3.12) is obtained formally from (3.11), since no account is taken of the characteristics of polynomial $D_p(s)$, which at certain $s = j\omega$ may vanish; however, the validity of (3.12) for all ω may be proved if (3.12) is obtained directly from Euler's equations for the extremals of solutions (1.1)-(1.3).

Assuming that in the control system with astatism the degree of astatism is determined by the controlled item, we can write

$$D_p(s) = s^r(d_0 + d_1 s + \ldots + d_m s^m) = s^r \bar{D}_p(s) d_0. \quad (3.13)$$

Substituting (3.13) into (3.12) and dividing each side of equality (3.12) by d_0, we obtain at $s_n = j\omega = 0$

$$K_p^2 = \frac{\sum_{k=1}^{n} L_k^2(0)}{d_0^2}. \quad (3.14)$$

Thus, if upon substitution of the solution of the synthesis, the values of the transfer coefficient K_p^* and cutoff frequency $\bar{\omega}_c$ are specified along with functional (1.3), then the coefficients q_{ij} $(i, j = 1, \ldots, n)$ must satisfy the conditions

$$\frac{1}{D_p(0)} \sum_{k=1}^{n} \left\{ \sum_{i=1}^{n} h_{ki} \left[\sum_{j=1}^{n} b_j r_{ij}(0) \right] \right\}^2 = K_p^{*2} \quad (3.15a)$$

for static systems,

$$\frac{1}{d_0^2} \sum_{k=1}^{n} \left\{ \sum_{i=1}^{n} h_{ki} \left[\sum_{j=1}^{n} b_j r_{ij}(0) \right] \right\}^2 = K_p^{*2} \quad (3.15b)$$

for astatic systems

$$\sum_{k=1}^{n} \left\{ \sum_{i=1}^{n} h_{ki} \left[\sum_{j=1}^{n} b_j r_{ij}(-j\bar{\omega}_c) \right] \right\} \left\{ \sum_{i=1}^{n} h_{ki} [b_j r_{ij}(j\bar{\omega}_c)] \right\} = 1, \quad (3.16)$$

and system (1.1) and (1.2), optimal in the sense of functional (1.3), after satisfying these conditions has K_p and ω_c near (to an accuracy determined above) to those specified.

We note that conditions (3.15) and (3.16) may turn out to be incompatible.

4. Example

Let us examine the gyro housing described by equations [6]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = p_{22}x_2 + p_{23}x_3, \quad \dot{x}_3 = p_{32}x_2 + p_{33}x_3 + b_3 u. \quad (4.1)$$

It is necessary to determine the control equation

$$u = c_1 x_1 + c_2 x_2 + c_3 x_3 \quad (4.2)$$

or, which is the same, the transfer function of the regulator

$$u = F(s) x_1 \quad (4.2')$$

so that, along the solutions (4.1) and (4.2) the functional of the form

$$I = \int_0^\infty (q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + u^2)\, dt, \quad (4.3)$$

is minimized, and in addition, system (4.1) and (4.2) should have K_p and $\bar{\omega}_c$ near certain given values K_p^* and $\bar{\omega}_c$.

We shall assume that in the functional (4.3) the coefficient q_{33} is specified. We determine q_{11} and q_{22} from the given values K_p and $\bar{\omega}_c$.

In the case under study

$$h_{11}^2 = q_{11}, \quad h_{22}^2 = q_{22}, \quad h_{33}^2 = q_{33},$$

$$D_p(s) = \begin{vmatrix} s & -1 & 0 \\ 0 & s - p_{22} & p_{23} \\ 0 & -p_{32} & s - p_{33} \end{vmatrix} = s\,[s^2 - (p_{22} + p_{33})\,s + p_{22}p_{33} - p_{23}p_{32}],$$

$$r_{13} = h_{11}p_{23}, \quad r_{23} = -h_{22}s\,p_{23}, \quad r_{33} = h_{33}s\,(s - p_{22}).$$

Conditions (3.15b) and (3.16) are of the form

$$\frac{q_{11}p_{23}^2\,b_3^2}{(p_{22}p_{33} - p_{32}p_{23})^2} = K_p^{*2}, \tag{4.4}$$

$$\frac{q_{11}p_{23}^2 b_3^2 + q_{22}\bar{\omega}_c^2 p_{23}^2 b_3^2 + q_{33}\bar{\omega}_c^2\,(\bar{\omega}_c^2 + p_{22}^2)\,b_3^2}{\bar{\omega}_c^2\left\{\bar{\omega}_c^4 + (p_{22}^2 + p_{23}^2 + 2p_{23}p_{32})\,\bar{\omega}_c^2 + (p_{22}p_{33} - p_{32}p_{23})^2\right\}} = 1. \tag{4.5}$$

The following gyro housing parameters [6] and the following optimization functional were chosen as a numerical example:

$$p_{22} = -300, \quad p_{23} = 10^3, \quad p_{32} = -3, \quad p_{33} = -1, \quad b_3 = 10^{-3},$$
$$q_{33} = 5\cdot 10^9, \quad K_p^* = 4\cdot 10^2 \left(K_p \approx \frac{K_{\text{reg gain}}}{H}\right), \tag{4.6}$$
$$\bar{\omega}_c = 100 - 200 \ \text{sec}^{-1}.$$

At $\bar{\omega}_c = 100$ it follows from (4.4) and (4.5) that

$$q_{11} = 1.6\cdot 10^{12}, \quad q_{22} = 3\cdot 10^8. \tag{4.7}$$

At numerical values of (4.6) and (4.7) of the gyro housing parameters and of the optimization functional, the problem of optimizing functional (4.3) was solved by the Lyapunov-Bellman method, and the coefficients c_i ($i = 1, 2, 3$) of the optimal regulator were obtained: $c_1 = -0.126\cdot 10^7$, $c_2 = -0.44\cdot 10^4$, and $c_3 = -116\cdot 10^6$.

Using the first and second equation of system (4.1), it is easy to obtain the transfer function of the optimal regulator:

$$u = c_1x_1 + \left(c_2 - c_3\frac{p_{22}}{p_{23}}\right)\dot{x}_1 + \frac{c_3}{p_{23}}\ddot{x}_1 = -0.126\cdot 10^7\,(0.92\cdot 10^{-4}s^2 + 3.1\cdot 10^{-2}s + 1)\,x_1.$$

The transfer function of the open state gyro housing is of the form

$$w(s) = \frac{p_{23}b_3\left[c_1 + \left(c_2 - c_3\frac{p_{22}}{p_{23}}\right)s + c_3s^2\right]}{s\,[s^2 - (p_{22} + p_{33})\,s + p_{22}p_{33} - p_{23}p_{32}]} = \frac{0.126\cdot 10^7\,(0.92\cdot 10^{-4}s^2 + 3.1\cdot 10^{-2}s + 1)}{s\,(s^2 + 301s + 3.3\cdot 10^3)}.$$

Figure 1 shows the automatic frequency response plots in the open and closed states.

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