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LINEAR SYSTEMS

Adaptive Controller for a Multi-Mode Object

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Abstract—We consider an adaptive controller for a single-mode object and study the possibility of using it for a multi-mode object. We propose an enhanced adaptive control algorithm for a multi-mode object. We study the influence of the quantization effect in DA and AD converters on the results of adaptive control. We study the influence of test signal amplitudes on identification results and show the findings of experimental studies for an adaptive controller with improved algorithms.

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1. INTRODUCTION

There are several research directions in the field of adaptive control theory where external disturbance is an unknown bounded function.

One of such directions is related to the notion of a reference model. Characteristic results of this direction can be found, for instance, in [1, 2]. The fundamental idea of this direction is easy to understand on the example of [1], where the authors solve an LQ-optimization problem for an object with unknown coefficients. Solving this problem with Riccati equations, they substitute quasiestimates constructed with the gradient method as true object coefficients. Such quasiestimates may significantly differ from true values under the influence of external disturbances. In [3], other possibilities this approach provides are discussed, possibilities that do not use quasiestimates.

One characteristic feature of the method of recurrent objective inequalities [4, 5] is the formulation of an adaptive control objective as admissible residues for the deviations of the established object output under the influence of arbitrary bounded disturbances. In [6, 7], solution of the l_1-optimization problem is generalized to the case of unknown object coefficients. In these works, the authors use a variation of the gradient method in such a way that the resulting quasiestimates for object coefficients ensure minimal deviation of the system output. However, numerical implementation for this method becomes more complicated.

Therefore, a number of works have been devoted to a more narrow class of external disturbances. For instance, in [8] the external disturbance is assumed to be a piecewise constant function with a given frequency range, while estimates of object coefficients can be found with a test signal and an adaptive observer.

In frequency adaptive control [9], the control objective is a given accuracy in object output, and the external disturbance can be an arbitrary bounded function. The method of finite–frequency identification [10] employs a test signal in the form of a finite sum of harmonics. Implementation of this approach began with the controller FAC-1 [11] and has undergone a number of modifications over the last two decades, including algorithms for tuning the amplitudes, frequencies, and duration of adaptation.

This work is organized as follows. In Section 2, we give the problem setting. Section 3 contains an exposition of known results for the problem's solution in the single-mode case and a description of the testbed. In Section 4, we describe the direct reconstruction method and propose a technique for suppressing noises caused by level quantization in the digital-analog (DA) and analog-digital (AD) converters together with a way to design an implementable controller with this property. In Section 5 we present numerical studies of the proposed controller that take into account the presence of a DA or AD converter in the system. In Section 6, we discuss the results of seminatural experimental studies. Appendices contain derivations of main expressions and the proof of the statement.

2. PROBLEM SETTING

Consider an object given by a difference equation

$$\begin{gathered} y(k)+d_1^{[m]}y(k-1)+\ldots+d_{n-1}^{[m]}y(k-n+1)+d_n^{[m]}y(k-n) \\ = k_1^{[m]}u(k-1)+\ldots+k_{n-1}^{[m]}u(k-n+1)+k_n^{[m]}u(k-n)+f(k), \\ k=0,1,2,\ldots, \quad m=1,2,\ldots, \end{gathered} \tag{2.1}$$

where $y(k)$ is the object output measured at time moment $t=kh$ (h is the discretization interval of this object); $u(k)$, the control; $f(k)$, unmeasured external disturbance which is an unknown bounded function ($|f(k)|\leqslant f^*$, where f^* is a given number); m, index of the object operation mode; n, a known number. Object coefficients $d_j^{[m]}$ and $k_j^{[m]}$ are unknown numbers that change at time moments $t_{\rm cm}^{[m]}$ and are constant on time intervals $\left[t_{\rm cm}^{[m]},t_{\rm cm}^{[m+1]}\right)$, $m=1,2,3,\ldots$, $t_{\rm cm}^{[1]}=0$. Time moments $t_{\rm cm}^{[m]}$, $m=2,3,4,\ldots$ are known or can be found in the adaptation process. The object is asymptotically stable on each of the operation modes m.

The control comes from a controller:

$$\begin{gathered} u(k)+g_1^{[m]}u(k-1)+\ldots+g_{\psi-1}^{[m]}u(k-\psi+1)+g_\psi^{[m]}u(k-\psi) \\ = r_0^{[m]}y_v(k-\psi+n-1)+r_1^{[m]}y_v(k-\psi+n-2)+\ldots+r_{n-1}^{[m]}y_v(k-\psi), \quad m=1,2,3,\ldots, \end{gathered} \tag{2.2}$$

where $y_v(k)\triangleq y(k)-v(k)$, $v(k)$ is the identifying signal, $r^{[m]}=[r_0^{[m]},\ldots,r_{n-1}^{[m]}]$, $g^{[m]}=[g_1^{[m]},\ldots,g_\psi^{[m]}]$ are controller coefficients, $\psi\geqslant n-1$ is a given number. Coefficients $r^{[m]}$, $g^{[m]}$ are found by the time moment $t_{\rm cm}^{[m]}+\Delta t_{\rm adapt}^{[m]}$, where $t_{\rm cm}^{[m]}$ is the moment when the mth operational mode begins at the object, $\Delta t_{\rm adapt}^{[m]}$ is the duration of adaptation on mode m. In the first mode (for $m=1$), object parameters are unknown, and as the control signal we give a test signal $u(k)=v(k)$. On subsequent modes on time intervals $\left[t_{\rm cm}^{[m]},t_{\rm cm}^{[m]}+\Delta t_{\rm adap}^{[m]}\right)$ controller (2.2) operates with coefficients $r^{[m-1]}$, $g^{[m-1]}$.

Transition function of the controller (2.2) has the following form:

$$w_{\rm cnt}(q)=\frac{r_0^{[m]}q^{\psi-n+1}+r_1^{[m]}q^{\psi-n+2}+\ldots+r_{n-1}^{[m]}q^\psi}{1+g_1^{[m]}q+\ldots+g_\psi^{[m]}q^\psi}, \tag{2.3}$$

where the shift operator q is defined as $q^ix(k)\triangleq x(k-i)$.

The problem is to find, for every $m=1,2,3,\ldots$, the controller coefficients $r^{[m]}$, $g^{[m]}$ such that controller (2.2) meets the following conditions on accuracy:

$$|y(k)|\leqslant y^\star, \quad k>k_\star^{[m]}, \tag{2.4}$$

where $y^\star$ is a given number, $k_\star^{[m]}$ is such that the time moment $t_\star=k_\star^{[m]}h$ lies inside the time interval $t_{\rm cm}^{[m]}<t<t_{\rm cm}^{[m+1]}$, $m=1,2,3,\ldots$. We make the following assumptions:

(a) there exists a number $k_\star^{[m]}$ satisfying (2.4);

(b) there exists a controller (2.2) that ensures that objective (2.4) is achieved for known object coefficients (2.1);

(c) the object coefficients change slowly from mode to mode, in such a way that the object in the mth mode with a controller constructed for the object in mode $(m-1)$ is still stable (the adjacent stability condition). However, if we do not change controller coefficients the system will lose stability in passing to some subsequent mode;

(d) the time interval until an object operation mode change exceeds the time interval needed for adaptation:

$$t_{\text{cm}}^{[m+1]} > t_{\text{cm}}^{[m]} + \Delta t_{\text{adap}}^{[m]}, \quad m = 1, 2, 3, \ldots,$$

where $\Delta t_{\text{adapt}}^{[m]}$ is the time needed for adaptation.

Experimental studies with a frequency adaptive controller whose results are shown in Sections 5 and 6 show a high level of noise in the control signal caused by level quantization in DA and AD converters, which leads to increased identification time and increased error in object parameter identification for the closed system. The purpose of this work is to further develop existing adaptive control algorithms to the case of a multimode object, update the algorithms with respect to the presence of DA and AD converters in the control system, and conduct experiments with the enhanced controller.

3. PRELIMINARIES

Let us begin by stating some results on solving the problem in the single-mode case [12] $(m = 1)$ and refining the problem by accounting for characteristic features caused by multiple modes in the object.

The structural scheme of an adaptive controller is shown on Fig. 1.

The test signal generator emits a test signal $v(k)$ that consists of n harmonics:

$$v(k) = \sum_{i=1}^{n} \rho_i \sin(\omega_i k h), \qquad 0 < \omega_i < \frac{\pi}{h},$$

where ρ_i is the amplitude of the ith harmonic and ω_i is its frequency.

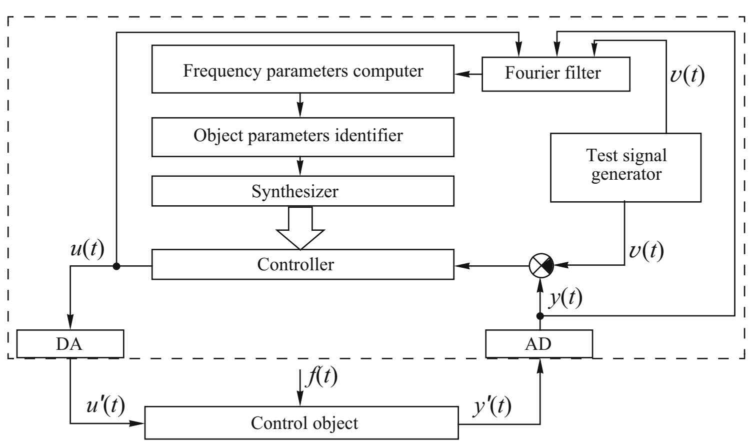

Fig. 1. Structural scheme of an adaptive controller.

The Fourier filter computes estimates $\hat{\alpha}_i,\ \hat{\beta}_i$ for frequency parameters [10] of the object:

$$\hat{\alpha}_{yi} = \alpha_{yi}(N) = \frac{2}{\rho_i N\tau}\sum_{k=k_0+1}^{k_0+N\tau} y(k)\sin(\omega_i kh), \quad \hat{\beta}_{yi} = \beta_{yi}(N) = \frac{2}{\rho_i N\tau}\sum_{k=k_0+1}^{k_0+N\tau} y(k)\cos(\omega_i kh),$$

$$\hat{\alpha}_{ui} = \alpha_{ui}(N) = \frac{2}{\rho_i N\tau}\sum_{k=k_0+1}^{k_0+N\tau} u(k)\sin(\omega_i kh), \quad \hat{\beta}_{ui} = \beta_{ui}(N) = \frac{2}{\rho_i N\tau}\sum_{k=k_0+1}^{k_0+N\tau} u(k)\cos(\omega_i kh),$$

$$N = 1, 2, 3, \ldots, \quad \tau = \frac{2\pi}{\min(\omega_i)h}.$$

Frequency parameters computer finds frequency parameters of the object:

$$\hat{\alpha}_i = \frac{\hat{\alpha}_{yi}\hat{\alpha}_{ui} + \hat{\beta}_{yi}\hat{\beta}_{ui}}{\hat{\alpha}_{ui}^2 + \hat{\beta}_{ui}^2}, \quad \hat{\beta}_i = \frac{-\hat{\alpha}_{yi}\hat{\beta}_{ui} + \hat{\beta}_{yi}\hat{\alpha}_{ui}}{\hat{\alpha}_{ui}^2 + \hat{\beta}_{ui}^2}, \quad i = \overline{1, n}.$$

Identifier solves frequency equations [10]:

$$\hat{k}(e^{-j\omega_i h}) - (\hat{\alpha}_i + j\hat{\beta}_i)\hat{d}(e^{-j\omega_i h}) = 0, \quad i = \overline{1, n},$$

$\hat{k}(q) = \hat{k}_1 q + \ldots + \hat{k}_{n-1}q^{n-1} + \hat{k}_n q^n$, $\hat{d}(q) = 1 + \hat{d}_1 q + \ldots + \hat{d}_{n-1}q^{n-1} + \hat{d}_n q^n$ are estimates of object polynomials $k(q) = k_1 q + \ldots + k_{n-1}q^{n-1} + k_n q^n$, $d(q) = 1 + d_1 q + \ldots + d_{n-1}q^{n-1} + d_n q^n$.

The transition function of the object found in identification can be described as

$$w(q) = \frac{\hat{k}(q)}{\hat{d}(q)}. \tag{3.1}$$

Object (3.1) represented in the state space has the following form:

$$\begin{cases} x(k) = Ax(k-1) + bu(k-1) \\ y(k) = cx(k), \end{cases} \tag{3.2}$$

where $x(k)$ is the n-dimensional vector of the object state; A, a matrix of size $n \times n$; b, an n-dimensional vector of numbers; c, a row of numbers.

The synthesizer solves the LQ-optimization problem with functional

$$J = \sum_{k=0}^{\infty}\Big\{x^T(k)Qx(k) + x_{n+1}^2(k) + \varepsilon_1^2\left[x_{n+2}(k)\right]^2 + \varepsilon_2^2\left[x_{n+3}(k)\right]^2 + \ldots + \varepsilon_{\psi-1}^2\left[x_{n+\psi}(k)\right]^2 + \varepsilon_\psi^2\left[\mu(k)\right]^2\Big\}, \tag{3.3}$$

where

$$Q = c^T\tilde{q}^2 c, \qquad \tilde{q} = \frac{f^*}{y^*},$$

$$x_{n+1}(k) \triangleq u(k), \quad x_{n+2}(k) \triangleq \frac{x_{n+1}(k+1) - x_{n+1}(k)}{h},$$

$$x_{n+3}(k) \triangleq \frac{x_{n+2}(k+1) - x_{n+2}(k)}{h}, \ \ldots, \ x_{n+\psi}(k) \triangleq \frac{x_{n+\psi-1}(k+1) - x_{n+\psi-1}(k)}{h},$$

$$\mu(k) \triangleq \frac{x_{n+\psi}(k+1) - x_{n+\psi}(k)}{h},$$

$\varepsilon_i^2 = \frac{C_\psi^i}{5^{2i}\omega_{\rm cp}^{2i}}$ $(i = \overline{1,\psi})$ are sufficiently small coefficients [13] that depend on the frequency of system slice $\omega_{\rm cp}$; here C_ψ^i is the binomial coefficient from ψ by i, and the number $\psi \geqslant n-1$ is found from the implementability condition of the controller's transition function. Note that for $h \to 0$ it holds that $x_{n+2} \to \dot{u}, x_{n+3} \to \ddot{u}, \ldots$.

The synthesizer finds controller coefficients in such a way that it satisfies conditions on control accuracy (2.4). It uses the estimates of object coefficients $\hat{k}_j$ and $\hat{d}_j,\ j = \overline{1,n}$, found through the experiments.

For an experimental study of FAC-25, a special testbed FM-2 has been developed [14]. The FM-2 testbed consists of an industrial controller WinCon W-8341 [15] that contains 14-bit DA and AD converters and an IBM-compatible single-board industrial computer Athena [16] with embedded 12-bit DA and 16-bit AD converters. FAC-25 operates on the industrial controller WinCon while the object is imitated by the industrial computer Athena.

4. SUPPRESSION OF NOISE (DISTURBANCES) CAUSED BY THE PRESENCE OF DA AND AD CONVERTERS

4.1. Basic Idea of Our Approach

To reduce the noises in the control signal we use the following approach. The essence of our idea is to increase the order of the denominator for the controller's transition function (2.3), i.e., it is a kind of filter for such disturbances (noises). We implement it as follows: in the quality functional (3.3) the order of controller's denominator is defined by the parameter $\psi = n-1$. We add to this value the number γ that leads to an increase in the order of controller's denominator: $\psi = n - 1 + \gamma$. We study how the controller's transition function depends on the number γ. To do so, we consider the primary stages of controller design.

4.2. Direct Method for Reconstructing the Phase Vector

Controller design is based on solving an LQ-optimization problem and constructing an observer with the direct reconstruction method [13].

The controller design algorithm is based on the optimal controller with respect to state in the sense of the following functional:

$$J = \sum_{k=0}^{\infty} \Big\{ x^T(k)Qx(k) + Ru^2(k) \Big\}, \tag{4.1}$$

where $Q \in \mathbb{R}^{n\times n},\ R = 1$ (for $\psi = 0$ functional (3.3) coincides with (4.1)).

Functional (4.1) is minimized on the motions of object (3.2) with state controller

$$u(k) = Kx(k), \tag{4.2}$$

where the n-dimensional row vector K is found via the solution of the Riccati equation.

Proposition. *We propose to perform controller design with state controller* (4.2) *and observer* (4.3):

$$q^\psi x(k) = l(q)u(k) + \lambda(q)y(k), \tag{4.3}$$

which implies that

$$[q^\psi - Kl(q)]u(k) = K\lambda(q)y(k). \tag{4.4}$$

Observer (4.3) results from the direct reconstruction method [13], and its derivation is given in Appendix A. In expression (4.3),

$$l(q) = -(L^0)^{-1}[l_1\ l_2\ \ldots l_{n-1}]\phi(q), \tag{4.5}$$

$$\lambda(q) = (L^0)^{-1}\psi(q), \tag{4.6}$$

where

$$L^0 = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix}, \quad l_1 = \begin{bmatrix} 0 \\ cb \\ cAb \\ \vdots \\ cA^{n-2}b \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 0 \\ cb \\ \vdots \\ cA^{n-3}b \end{bmatrix}, \quad \ldots, \quad l_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ cb \end{bmatrix}, \tag{4.7}$$

$$\phi(q) = \begin{bmatrix} q^\psi \\ q^{\psi-1} \\ q^{\psi-2} \\ \vdots \\ q^{\psi-n+2} \end{bmatrix}, \quad \psi(q) = \begin{bmatrix} q^\psi \\ q^{\psi-1} \\ q^{\psi-2} \\ \vdots \\ q^{\psi-n+1} \end{bmatrix}, \tag{4.8}$$

where L^0 is an $n \times n$ matrix, $l_1, \ldots, l_{n-1}$ are n-dimensional vectors.

In [13], an observer similar to (4.3) has been constructed with the use of "past" values of the signal $y(k)$, $u(k)$, i.e., values that are preserved in the memory of the implemented controller's computer. This approach has the following disadvantage: in constructing the observer we use the inverse matrix A^{-1} which may not exist if there is an integrator (a zero eigenvalue in the matrix A) in the object. The proposed approach has no such disadvantage.

The transition function of the controller (4.4) can be written as

$$w_{\text{cnt}}(q) = \frac{K\lambda(q)}{(q^\psi - Kl(q))}. \tag{4.9}$$

Note that in expression (4.9) polynomials $(K\lambda(q))$ and $(q^\psi - Kl(q))$ have their largest degree equal to ψ; the least degree in the numerator $(q^\psi - Kl(q))$ equals $\psi - n + 1$, in the denominator, $(K\lambda(q)) - \psi - n + 2$. This implies that transition function (4.9), generally speaking, cannot be implemented. In order to implement a controller we will consider the functional (3.3).

4.3. A Method for Amplifying the Controller's Filtering Properties. Designing an Implementable Controller

As we have already noted, experimental studies with a frequency adaptive controller have shown a high level of noises in the control signal caused by level quantization in DA and AD converters which has led to increased duration and errors in the object's identification. The basic idea for suppressing such noises (disturbances) is to increase the order in the denominator of the controller's transition function. This can be done as follows.

The controller in question is designed with the functional (3.3) with parameter $\psi = n - 1$ [13]. In order to amplify filtering properties, the controller's parameter ψ can be increased by γ: $\psi = n - 1 + \gamma$. Solving the problem defined in Section 4.2 with functional (3.3) leads to a state controller that has the following form:

$$\mu(k) = \overline{K}\overline{x}(k); \tag{4.10}$$

its derivation is given in Appendix B. Expression (4.10) can be rewritten as

$$\tilde{d}(q)u(k) = K^{[1]}q^{\psi}x(k), \tag{4.11}$$

where $\tilde{d}(q) = q^{\psi}\frac{\det(Iq^{-1}-\tilde{A})^{-1}}{h^{\psi}}$ is a polynomial of degree ψ, matrix $\tilde{A} \in \mathbb{R}^{\psi\times\psi}$ has the following structure:

$$\tilde{A} = \begin{bmatrix} 1 & h & 0 & 0 & \dots & 0 \\ 0 & 1 & h & 0 & \dots & 0 \\ 0 & 0 & 1 & h & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & h \\ hK^{[2]}(1) & hK^{[2]}(2) & hK^{[2]}(3) & hK^{[2]}(4) & \dots & hK^{[2]}(\psi)+1 \end{bmatrix}, \tag{4.12}$$

$K^{[1]} \in \mathbb{R}^{1\times n}$, $K^{[2]} \in \mathbb{R}^{1\times\psi}$ are rows of numbers that result from optimizing functional (3.3) with an extended state vector.

Substituting (4.3) into (4.11), we get an implementable controller in the following form:

$$\left(\tilde{d}(q) - K^{[1]}l(q)\right)u(k) = K^{[1]}\lambda(q)y(k). \tag{4.13}$$

Derivations of expressions (4.11), (4.13) are given in Appendix B.

Let us study the stability of system (3.2), (4.13). To do so, we compare characteristic polynomials of systems (3.2), (4.11) and (3.2), (4.13). Consider the characteristic polynomial of system (3.2), (4.11), which is asymptotically stable by construction. Using the formula for the determinant of a 2×2 block matrix, we get

$$D(q) = \begin{vmatrix} I - Aq & -bq \\ -q^{\psi}K^{[1]} & \tilde{d}(q) \end{vmatrix} = |I - Aq|\left(\tilde{d}(q) - q^{\psi+1}K^{[1]}(I - Aq)^{-1}b\right), \tag{4.14}$$

where I is the unit matrix of the corresponding dimension.

Similarly, we write the characteristic polynomial of the closed system (3.2), (4.13):

$$D^{(1)}(q) = \begin{vmatrix} I - Aq & -bq \\ -K^{[1]}\lambda(q)c & \tilde{d}(q) - K^{[1]}l(q) \end{vmatrix}$$
$$= |I - Aq|\left(\tilde{d}(q) - K^{[1]}l(q) - K^{[1]}\lambda(q)c(I - Aq)^{-1}bq\right). \tag{4.15}$$

Statement. *Characteristic polynomials of systems without observer $D(q)$ and with observer $D^{(1)}(q)$ coincide: $D^{(1)}(q) = D(q)$. Since $D(q)$ is stable by construction (since the controller results from optimizing a quadratic functional (3.3)), this means that $D^{(1)}(q)$ is also stable.*

The proof of statement for the case $n = 3$ is given in Appendix C.

In controller (4.13), the largest degree in the left- and right-hand sides equals $\deg(\tilde{d}(q)) = \psi$. The smallest degree in the left-hand side (denominator of the controller's transition function) is zero, and the smallest degree of the right-hand side (numerator of the controller's transition function) equals $\psi - (n - 1) = \gamma$. This implies that the value of parameter γ may increase the order in the controller's transition function denominator. As we will see below, increasing the order of the controller's denominator by γ lets us efficiently suppress undesirable noises (disturbances) in the control signal caused by level quantization in DA and AD converters.

5. MODELING A SYSTEM WITH DA AND AD CONVERTERS

5.1. Quantization Model for DA, AD

It is known [17] that level quantization in DA and AD converters means that the signal has a finite number of possible states with respect to the level; the number of states is defined by the capacity of a DA or AD converter. The number of these states is given by $M = 2^p$, where p is the capacity and M is the number of possible values. The quantization step δ is given by $\delta = 2V/2^p$, where V determines the limit value of a quantizer; e.g., for the FM-2 testbed $V = 10$V. A perfect quantizer characteristic has the form shown on Fig. 2.

Following [17], we write this characteristic, taking into account its asymmetry due to the odd number of "steps," as

$$g_1 = \begin{cases} V-\delta & \text{for} \quad g > V-\delta \\ g & \text{for} \quad -V \leqslant g \leqslant V-\delta \\ -V & \text{for} \quad g < -V; \end{cases} \tag{5.1}$$

$$\tilde{g} = \frac{2V}{2^p} \left\lfloor (g_1 + \delta/2)\, \frac{2^p}{2V} \right\rfloor, \tag{5.2}$$

where g is a signal continuous with respect to the level, g_1 is the signal g after the saturation of quantizer characteristic, brackets $\lfloor \ldots \rfloor$ denote rounding down, and the expression (5.2) itself defines level quantization with respect to the signal g_1.

In modeling an object identification process in a closed system, we need to take into account the level of quantization for DA and AD converters on each of the discrete time ticks.

We write controller (2.2) as equations in the state space

$$\begin{cases} x_c(k) = A_c x_c(k-1) + b_c y_v(k-1) \\ u(k) = c_c x_c(k) + d_c y_v(k-1). \end{cases}$$

For the modeling we have constructed the following algorithm. For every time moment k, signals of object output $y(k)$ and control $u(k)$ that have passed through DA and AD converter quantizers are given by expression

$$\tilde{y}(k) = f_{\mathrm{DA}}(y(k)), \quad \tilde{\tilde{y}}(k) = f_{\mathrm{AD}}(\tilde{y}(k)),$$
$$\tilde{u}(k) = f_{\mathrm{DA}}(u(k)), \quad \tilde{\tilde{u}}(k) = f_{\mathrm{AD}}(\tilde{u}(k)).$$

Here f_{AD} and f_{DA} denote the functions of AD- and DA-transformation according to algorithm (5.1) and (5.2) that assign a value $\tilde{g}$ to a continuous signal g.

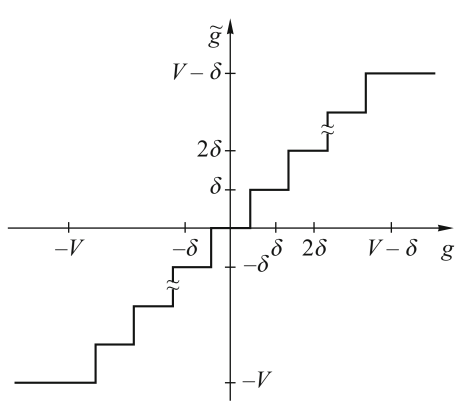

Fig. 2. Static characteristic of the quantizer.

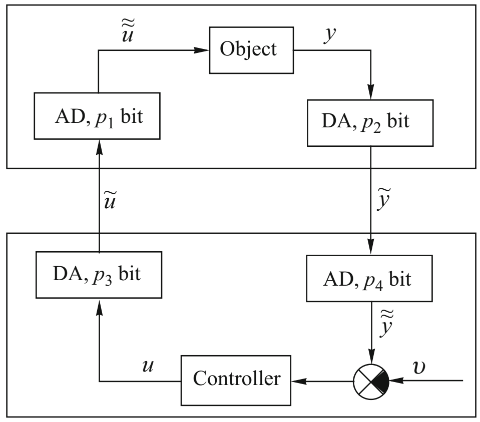

Fig. 3. The testbed structure with DA and AD converters.

Each of the signals $y(k),\ u(k)$ is quantized with the capacity of DA and AD converters according to Fig. 3. The experimental testbed FM-2 has $p_1 = 16$, $p_2 = 12$, $p_3 = 14$, $p_4 = 14$ bit.

After DA and AD converter quantizers, the object and controller will be given by

$$\begin{cases} x(k) = Ax(k-1) + b\tilde{\tilde{u}}(k-1) \\ y(k) = cx(k), \end{cases}$$

$$\begin{cases} x_c(k) = A_c x_c(k-1) + b_c(v(k-1) - \tilde{\tilde{y}}(k-1)) \\ u(k) = c_c x_c(k) + d_c(v(k-1) - \tilde{\tilde{y}}(k-1)). \end{cases}$$

5.1.1. Numerical experiment 1. A continuous counterpart of the object model has the following form:

$$w_{\text{o}}(s) = \frac{0.4s + 1}{0.2s^3 + 1.24s^2 + 5.24s + 1}.$$

The controller is given by transition function

$$w_{\text{r}}(q) = \frac{-80.37 + 154.4q - 74.24q^2}{1 - 1.655q + 0.6741q^2}.$$

Controller design is done with the following values of parameters: $\gamma = 0$, $\tilde{q} = 10$, $\varepsilon_1 = 9.35\times10^{-6}$, $\varepsilon_2 = 0.006$.

Test signal:

$$v(t) = 0.05\sin(0.2t), \tag{5.3}$$

external disturbance $f(t) = 0$.

Figures 4 and 5 show the influence of level quantization on the operation of a closed system. Figures 4a and 4b show respectively the object output and control signal without taking level quantization into account. Figures 5a and 5b show respectively the object output and control signal with level quantization in DA and AD converters.

The figures indicate that due to level quantization in a closed system the control signal is under a lot of noise, which leads to imprecise object identification in the closed system.

5.1.2. Numerical experiment 2. Let us see what happens as we increase the order of the controller's denominator for $\gamma = 2$. The controller with an extended denominator is defined by the

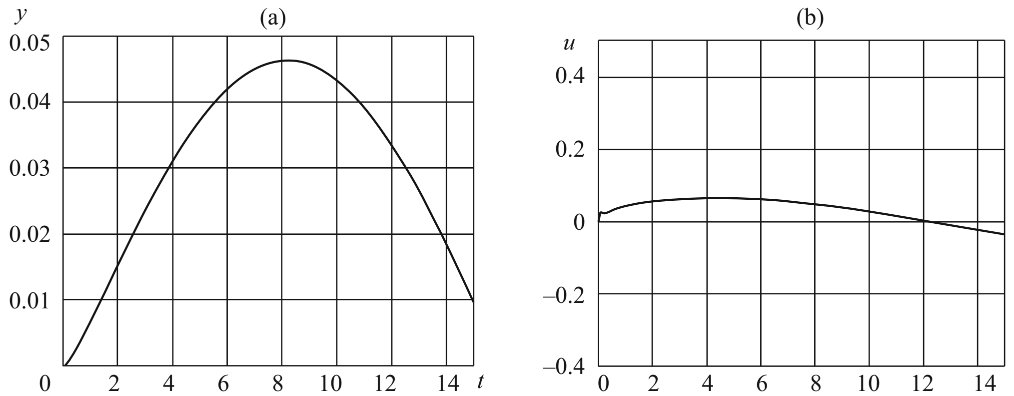

Fig. 4. Signals y, u without level quantization.

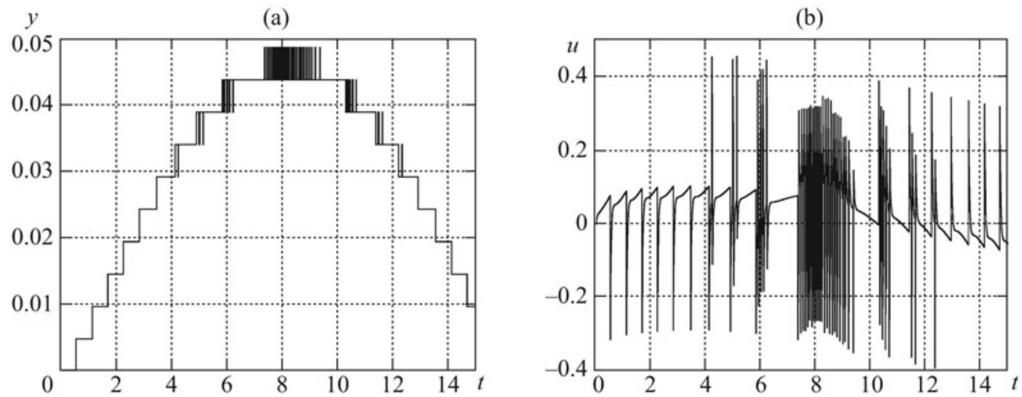

Fig. 5. Signals y, u with level quantization.

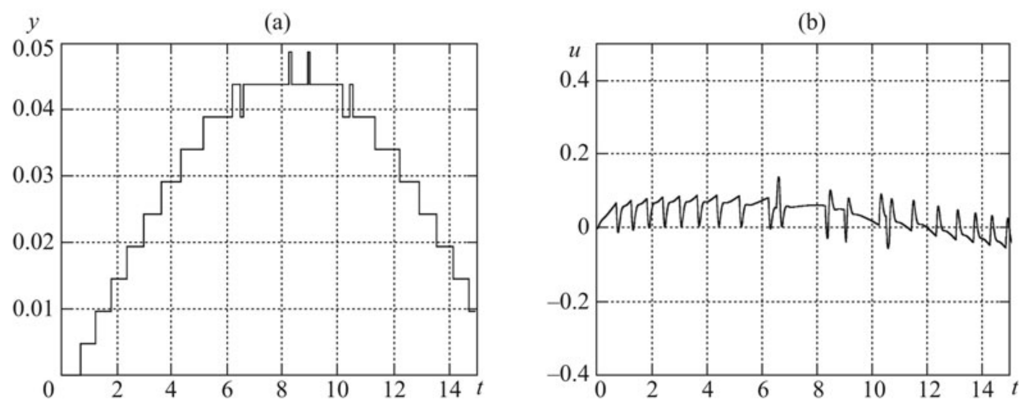

Fig. 6. Signals y, u with controller (5.4).

transition function

$$w_{\rm cnt}(q) = \frac{-2.532q^2 + 4.882q^3 - 2.357q^4}{1 - 3.318q + 4.132q^2 - 2.288q^3 + 0.4752q^4}. \tag{5.4}$$

Controller design has been done for the following values of parameters: $\tilde{q} = 10$, $\varepsilon_1 = 0.012$, $\varepsilon_2 = 5.61 \times 10^{-5}$, $\varepsilon_3 = 1.14 \times 10^{-7}$, $\varepsilon_4 = 8.75 \times 10^{-11}$.

As the test signal we used function (5.3), $f(t) = 0$.

Figures 6a and 6b show respectively the object output $y(t)$ and control signal $u(t)$ with the effect of level quantization as a result of modeling with the controller (5.4).

The figures indicate that by increasing the order of the controller's transition function denominator we can effectively reduce the noises cause by quantization in DA and AD converters.

5.2. The Influence of Test Signal Amplitude on Identification Result in a Closed System

The presence of level quantization in a control system imposes constraints on test signal amplitudes. Obviously, a small test signal amplitude, less than one step of quantization, will not have any influence on the system's output, so it will be impossible to identify the object. The question arises: for what values of test signal amplitudes will the identification result not be significantly influenced by level quantization?

To answer this question, we have studied the influence of test signal amplitudes on identification results. By the identification result we mean that the coefficients of the original and identified

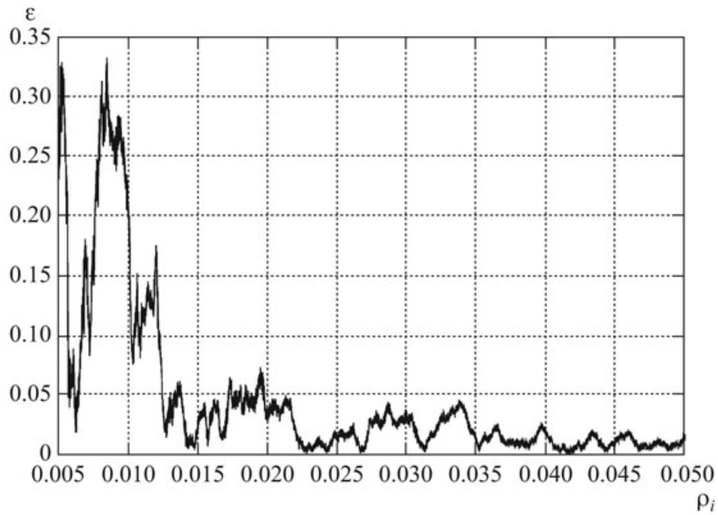

Fig. 7. Dependence of relative error in the coefficients of the identified object on the test signal amplitudes.

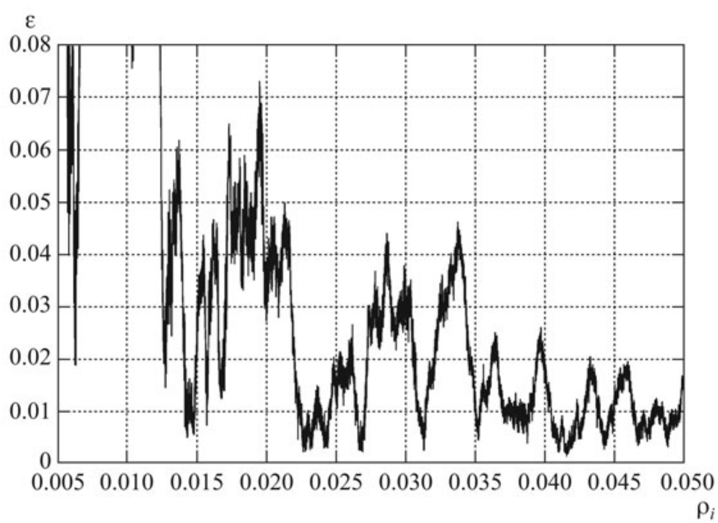

Fig. 8. Dependence of relative error in the coefficients of the identified object on the test signal amplitudes (enlarged).

objects are close in the following sense:

$$\varepsilon_{\text{ident}} = \max_i \left(\frac{\hat{k}_i - k_i}{k_i}, \frac{\hat{d}_i - d_i}{d_i} \right), \quad i = \overline{0, n}, \tag{5.5}$$

where k_i, d_i are coefficients in the continuous counterpart of object (2.1); $\hat{k}_i$, $\hat{d}_i$, estimates of coefficients k_i, d_i found in identification.

Figures 7 and 8 show numerical modeling results for the identification process. The horizontal axis shows test signal amplitudes ρ_i (identical for all harmonics); the vertical axis, identification error (5.5).

For small test signal amplitudes, quantization does have a significant effect on identification results. In general, this dependence has an exponential form: as test signal amplitudes increase the identification error decreases. As test signal amplitudes grow, the relative identification error stabilizes on the level of 1–3%.

6. EXPERIMENTAL STUDIES

The FM-2 testbed described in Section 3 has been used for experimental studies of a FAC-25 controller for a multi-mode object.

To test an adaptive controller in a closed system, we have generated 16 modes with the corresponding object coefficients at random. We assume that the object is given by

$$y(s) = \frac{K_{\text{obj}}(T_1 s + 1)}{(T_2 s + 1)(T_3^2 s^2 + 2T_3\xi + 1)}\, u(s) + \frac{1}{(T_2 s + 1)(T_3^2 s^2 + 2T_3\xi + 1)} f(s),$$
$$T_1 = 0.4, \quad \xi = 0.6.$$

Object parameters for the 16 modes are given in Tables 1 and 2.

As an external disturbance acting on the object we have chosen the function $f(t) = \text{sgn}(\sin(1.3t))$.

Note that the system with the first controller without adaptation loses stability in mode 16 when the object changes its operation modes.

Experimental results are shown in Tables 3 and 4 and on Figs. 9 and 10. Tables 3 and 4 show maximal deviations of the object output after the adaptation process is over in each mode (stage 3 on Figs. 9 and 10). In the experiments, the control objective (2.4) was specified as the number $y^\star = 0.1$. The tables indicate that for all modes the control objective (2.4) is satisfied. Experiments were conducted as follows. Before adaptation, the object was working in mode m, and the controller was taken from the object working in mode $m-1$. The adaptation process is turned on by an operator's demand. After the adaptation process is over, in a certain time the object changes

Table 1. Object modes 1–8

Mode number	K_{obj}	T_2	T_3
1	1	5	0.2
2	1.32	6.99	0.15
3	0.93	10	0.19
4	1.32	13.91	0.26
5	0.92	19.86	0.19
6	1.22	13.47	0.14
7	1.75	9.44	0.19
8	2.47	12.52	0.28

Table 2. Object modes 9–16

Mode number	K_{obj}	T_2	T_3
9	1.68	8.4	0.21
10	1.24	5.93	0.16
11	1.65	7.89	0.2
12	1.24	5.39	0.14
13	1.36	4.47	0.48
14	1.45	3.5	0.65
15	2.6	3.12	1.2
16	4.25	4.73	1.14

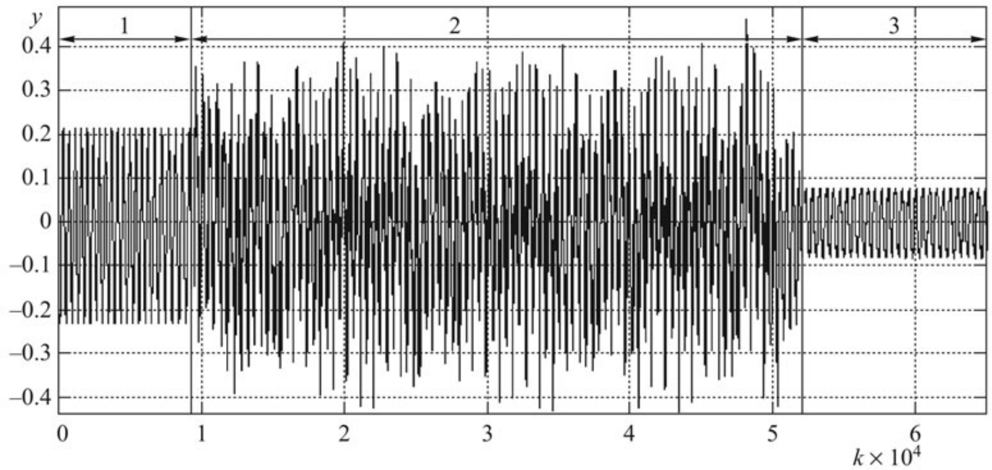

Fig. 9. Object output $y(k)$ in mode 1.

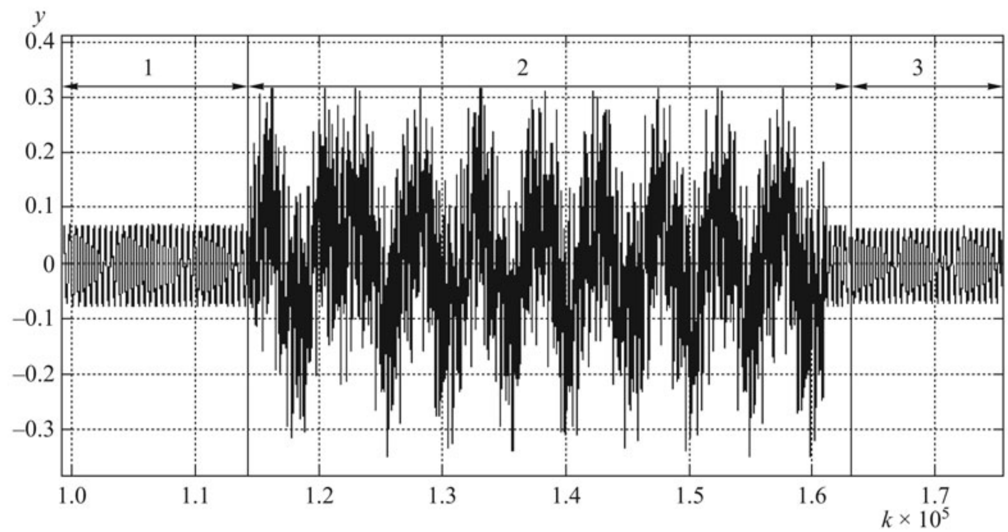

Fig. 10. Object output $y(k)$ in mode 10.

its operational mode, also by request of an operator. The adaptation process includes object identification and controller synthesis according to the algorithm described in Sections 3 and 4.

Figures 9 and 10 show experimental results corresponding to object modes 1 and 10. The numbers on these figures denote the following stages: 1, object output before adaptation; 2, the adaptation stage and computing a new controller; 3, system operation after adaptation with the new controller.

Table 3. Experimental results on an FM-2 testbed in modes 1–8

Index	1	2	3	4	5	6	7	8
$\max \lvert y(k) \rvert$	0.08	0.06	0.05	0.05	0.03	0.04	0.05	0.04

Table 4. Experimental results on an FM-2 testbed in modes 9–16

Index	9	10	11	12	13	14	15	16
$\max \lvert y(k) \rvert$	0.05	0.07	0.05	0.06	0.08	0.05	0.05	0.04

7. CONCLUSION

Our main results are the following:

- we have developed a frequency adaptive controller for a multi-mode object;
- we have proposed a method for constructing a controller with observer that uses "future" values $y(k)$, $u(k)$ and shifts by ψ ticks back, since the method for observer construction described in [13] uses an inverse object matrix that may not exist;
- we have studied the effect of level quantization in DA and AD converters on identification results and adaptive controls;
- we have proposed a way to amplify the controller's filtering properties that decreases the influence of noises caused by DA and AD converters quantization.

The proposed algorithms have been proven efficient with experimental studies on a multi-mode object.

ACKNOWLEDGMENTS

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APPENDIX A

Direct observer construction. Using object Eqs. (3.2), we write n equations of the form

$$\begin{cases} y(k-\psi) = cx(k-\psi) \\ y(k-\psi+1) = cx(k-\psi+1) = c(Ax(k-\psi) + bu(k-\psi)) \\ y(k-\psi+2) = cx(k-\psi+2) = c(A^2x(k-\psi) + Abu(k-\psi) + bu(k-\psi+1)) \\ \vdots \\ y(k-\psi+(n-1)) = c(A^{n-1}x(k-\psi) + A^{n-2}bu(k-\psi) + \ldots + bu(k-\psi+(n-2))) \end{cases}$$

or in matrix form:

$$\overline{y} = L^0x(k-\psi) + l_1u(k-\psi) + l_2u(k-\psi+1) + \ldots + l_{n-1}u(k-\psi+(n-2)), \tag{A.1}$$

where L^0, $l_1, \ldots, l_{n-1}$ have been defined in (4.7), $\overline{y} = \begin{bmatrix} y(k-\psi) \\ y(k-\psi+1) \\ y(k-\psi+2) \\ \vdots \\ y(k-\psi+(n-1)) \end{bmatrix}$, $L^0 \in \mathbb{R}^{n\times n}$, $l_1, \ldots, l_{n-1}$ are n-dimensional vectors.

Equation (A.1) can be rewritten as

$$\psi(q)y(k) = q^{\psi}L^0x(k) + [l_1\ l_2\ \ldots l_{n-1}]\phi(q)u(k),$$

where $\psi(q)$, $\phi(q)$ have been defined in (4.8).

This implies that

$$q^{\psi}x(k) = l(q)u(k) + \lambda(q)y(k),$$

where $l(q) = -(L^0)^{-1}[l_1\ l_2\ \ldots l_{n-1}]\phi(q)$, $\lambda(q) = (L^0)^{-1}\psi(q)$.

APPENDIX B

Controller's synthesis stages. On the first stage, we synthesize a state controller (4.2) for object (3.2). After that, the resulting controller is subject to the direct reconstruction method (Section 4.2), thus obtaining an output controller

$$[q^{\psi} - Kl(q)]u(k) = K\lambda(q)y(k),$$

in which polynomials $(K\lambda(q))$ and $(q^{\psi} - Kl(q))$ have largest degree ψ, the least degree in the numerator $(q^{\psi} - Kl(q))$ equals $\psi - n + 1$, in the denominator $(K\lambda(q))$, $\psi - n + 2$, i.e., such a controller cannot be physically implemented.

On the second stage we find the slice frequency [13] for system ω_{cp}.

On the third stage we compose an extended system. To achieve physical implementability and amplify the controller's filtering properties, we introduce additional terms in the functional (3.3): $x_{n+1} \dots x_{n+\psi}$ with coefficients ε_i $(i = \overline{1, \psi},\ \psi = n - 1 + \gamma)$.

Let us further consider the newly introduced functional terms $x_{n+1} \dots x_{n+\psi}$ as additional elements of the state vector $\overline{x}$ for some extended system

$$\overline{x}(k) = \Phi\overline{x}(k-1) + p\mu(k-1), \tag{B.1}$$

where

$$\overline{x}(k) = \begin{bmatrix} x(k) & \tilde{x}(k) \end{bmatrix}^T, \quad \tilde{x}(k) = \begin{bmatrix} x_{n+1}(k) & x_{n+2}(k) & \dots & x_{n+\psi}(k) \end{bmatrix}^T,$$
$$\Phi \in \mathbb{R}^{(n+\psi)\times(n+\psi)}, \quad p \in \mathbb{R}^{n+\psi},$$

and the term $\mu(k)$ is a control signal in this extended system:

$$\mu(k) \triangleq \frac{x_{n+\psi}(k+1) - x_{n+\psi}(k)}{h}.$$

Thus, we pose the optimal control problem for system (B.1) in the sense of the following functional:

$$\overline{J} = \sum_{k=0}^{\infty} \left\{ \overline{x}^T(k)\overline{Q}\overline{x}(k) + \overline{R}\mu^2(k) \right\}, \tag{B.2}$$

which coincides (up to notation) with functional (4.1).

Using the notation from (3.3), we write expression

$$\begin{cases} x_{n+1}(k+1) \doteq hx_{n+2}(k) + x_{n+1}(k) \\ x_{n+2}(k+1) \doteq hx_{n+3}(k) + x_{n+2}(k) \\ \dots \\ x_{n+\psi-1}(k+1) \doteq hx_{n+\psi}(k) + x_{n+\psi-1}(k) \\ x_{n+\psi}(k+1) \doteq x_{n+\psi}(k) + h\mu(k). \end{cases} \tag{B.3}$$

Taking into account object Eq. (3.2) and expressions (B.3), the extended control system (B.1) can be written as

$$\begin{cases} x(k) = Ax(k-1) + bx_{n+1}(k-1) \\ x_{n+1}(k) = hx_{n+2}(k-1) + x_{n+1}(k-1) \\ x_{n+2}(k) = hx_{n+3}(k-1) + x_{n+2}(k-1) \\ \dots \\ x_{n+\psi-1}(k) = hx_{n+\psi}(k-1) + x_{n+\psi-1}(k-1) \\ x_{n+\psi}(k) = x_{n+\psi}(k-1) + h\mu(k-1). \end{cases} \tag{B.4}$$

Taking into account (B.4), matrices of system (B.1) in block form look like

$$\Phi = \begin{bmatrix} A & B \\ 0_n^{\psi} & G \end{bmatrix}, \quad p = \begin{bmatrix} 0_1^{n+\psi-1} \\ h \end{bmatrix},$$

where $\Phi \in \mathbb{R}^{(n+\psi)\times(n+\psi)}$, $p \in \mathbb{R}^{(n+\psi)\times 1}$, 0_n^{ψ} denotes the zero matrix of size $\psi \times n$, $B = \begin{bmatrix} b & 0_{\psi-1}^{n} \end{bmatrix} \in \mathbb{R}^{n\times\psi}$, $G = \begin{bmatrix} 1 & h & 0 & \dots & 0 & 0 \\ 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{\psi\times\psi}$, and matrices $\overline{Q}, \overline{R}$ of the functional (B.2) look like

$$\overline{Q} = \begin{bmatrix} Q & 0_{\psi}^{n} \\ 0_n^{\psi} & F \end{bmatrix}, \quad \overline{R} = \varepsilon_{\psi}^2, \tag{B.5}$$

where $Q \in \mathbb{R}^{n\times n}$, $F = \operatorname{diag}(1, \varepsilon_1^2, \dots, \varepsilon_{\psi-1}^2)$.

On the fourth stage we look for optimal control for system (B.1) in the sense (B.2)

$$\mu(k) = \overline{K}\overline{x}(k), \tag{B.6}$$

with a procedure described in Section 4.2 but using matrices (B.5) instead of matrices Q, R of the functional (4.1).

On the fifth stage, we compose the controller equation. To do so, we rewrite Eq. (B.6) as

$$\mu(k) = \tilde{u}(k) + K^{[2]}\tilde{x}(k), \quad \tilde{u}(k) \triangleq K^{[1]}x(k). \tag{B.7}$$

Then $\overline{K}$ will be written as $[K^{[1]}K^{[2]}] \doteq \overline{K}$, where $K^{[1]}$ is a string of length n, and $K^{[2]}$ is a string of length ψ.

Note that it holds that

$$u(k) = \tilde{c}\tilde{x}(k), \tag{B.8}$$

where $\tilde{c} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{\psi}$.

Taking into account (B.7), we can write (B.3) as

$$\begin{cases} x_{n+1}(k+1) = hx_{n+2}(k) + x_{n+1}(k) \\ x_{n+2}(k+1) = hx_{n+3}(k) + x_{n+2}(k) \\ \dots \\ x_{n+\psi-1}(k+1) = hx_{n+\psi}(k) + x_{n+\psi-1}(k) \\ x_{n+\psi}(k+1) = x_{n+\psi}(k) + hK^{[2]}\tilde{x}(k) + h\tilde{u}(k), \end{cases}$$

or even in a more compact form due to (B.8):

$$\begin{cases} \tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{b}\tilde{u}(k) \\ u(k) = \tilde{c}\tilde{x}(k), \end{cases} \tag{B.9}$$

where matrix $\tilde{A}$ has been defined in (4.12), $\tilde{b} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & h \end{bmatrix}^T \in \mathbb{R}^{\psi}$.

To get controller equation, we write an expression that relates $\tilde{u}(k)$ and $u(k)$ in system (B.9):

$$u(k) = \tilde{c}(Iz - \tilde{A})^{-1}\tilde{b}\tilde{u}(k), \tag{B.10}$$

where z is the shift operator: $z^i x(k) \triangleq x(k+i)$.

We denote

$$\tilde{d}(z) \triangleq \det(Iz - \tilde{A}), \tag{B.11}$$

$$\xi(z) \triangleq \det(Iz - \tilde{A})\tilde{c}(Iz - \tilde{A})^{-1}\tilde{b}. \tag{B.12}$$

Let us show that $\xi(z) = h^{\psi}$. We denote

$$\tilde{H} = Iz - \tilde{A}. \tag{B.13}$$

Matrix $\tilde{H} \in \mathbb{R}^{\psi\times\psi}$ has the structure

$$\tilde{H} = \begin{bmatrix} z-1 & -h & 0 & 0 & \dots & 0 \\ 0 & z-1 & -h & 0 & \dots & 0 \\ 0 & 0 & z-1 & -h & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -h \\ -hK^{[2]}(1) & -hK^{[2]}(2) & -hK^{[2]}(3) & -hK^{[2]}(4) & \dots & z-hK^{[2]}(\psi)-1 \end{bmatrix}.$$

Transforming (B.12), with (B.13) we get that

$$\xi(z) \doteq \det\tilde{H} \times \tilde{c}\tilde{H}^{-1}\tilde{b} = \tilde{c}\begin{bmatrix} \tilde{H}_{1,1} & \dots & -1^{1+\psi}\tilde{H}_{1,\psi} \\ \vdots & \ddots & \vdots \\ -1^{\psi+1}\tilde{H}_{\psi,1} & \dots & \tilde{H}_{\psi,\psi} \end{bmatrix}^T \tilde{b}$$
$$= \begin{bmatrix} \tilde{H}_{1,1} & \dots & -1^{\psi+1}\tilde{H}_{\psi,1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & h \end{bmatrix}^T = -1^{\psi+1}\tilde{H}_{\psi,1}h,$$

where $-1^{i+j}\tilde{H}_{i,j}$ is the algebraic complement for an element of matrix $\tilde{H}$ at the intersection of row i and column j. Therefore,

$$\xi(z) = -1^{\psi+1}\tilde{H}_{\psi,1}h = -1^{\psi+1}h\begin{vmatrix} -h & 0 & 0 & \dots & 0 \\ z-1 & -h & 0 & \dots & 0 \\ 0 & z-1 & -h & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z-1 & -h \end{vmatrix}. \tag{B.14}$$

Expanding the determinant in (B.14) $\psi - 1$ times by the first rows, we get

$$\xi(z) = -1^{\psi+1}\tilde{H}_{\psi,1}h = -1^{1+\psi}h(-1^{\psi-1})h^{\psi-1} = (-1)^{2\psi}h^{\psi} = h^{\psi}. \tag{B.15}$$

By construction, the order $\deg(\tilde{d}(z)) = \dim(\tilde{A}) = \psi$.

Taking into account (B.11), (B.12), and (B.15), we can rewrite (B.10) as

$$\tilde{d}(z)u(k) = h^{\psi}K^{[1]}x(k). \tag{B.16}$$

Denoting $\check{d}(z) \triangleq \frac{\tilde{d}(z)}{h^{\psi}}$ and multiplying both parts of (B.16) by $z^{-\psi}$, we get (4.11):

$$\tilde{d}(q)u(k) = K^{[1]}q^{\psi}x(k), \tag{B.17}$$

where $\tilde{d}(q) \triangleq z^{-\psi}\check{d}(z)$.

Adding to (B.17) the observer Eq. (4.3) $q^{\psi}x(k) = l(q)u(k) + \lambda(q)y(k)$, we get (4.13):

$$\left(\tilde{d}(q) - K^{[1]}l(q)\right)u(k) = K^{[1]}\lambda(q)y(k).$$

APPENDIX C

Proof of Statement. For brevity we show the proof for $n = 3$. Consider the following expression:

$$q^{\psi+1}(I - Aq)^{-1}b = q^{\psi+1}(I - Aq)^{-1}b - l(q) + l(q) \qquad \text{(C.1)}$$
$$= l(q) + (L^0)^{-1}L^0\left(q^{\psi+1}(I - Aq)^{-1}b - l(q)\right).$$

We transform the expression $L^0\left(q^{\psi+1}(I - Aq)^{-1}b - l(q)\right)$. Consider expressions (4.5) and (4.8) that for $n = 3$ are written as $\psi(q) = \begin{bmatrix} q^{\psi} \\ q^{\psi-1} \\ q^{\psi-2} \end{bmatrix}$, $l(q) = -(L^0)^{-1}(l_1 q^{\psi} + l_2 q^{\psi-1})$. Using (4.7) we get that

$$L^0\left(q^{\psi+1}(I - Aq)^{-1}b - l(q)\right) = L^0 q^{\psi+1}(I - Aq)^{-1}b + l_1 q^{\psi} + l_2 q^{\psi-1}$$
$$= \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} q^{\psi+1}(I - Aq)^{-1}b + \begin{bmatrix} 0 \\ cb \\ cAb \end{bmatrix} q^{\psi} + \begin{bmatrix} 0 \\ 0 \\ cb \end{bmatrix} q^{\psi-1}$$
$$= \begin{bmatrix} q^{\psi+1}c(I - Aq)^{-1}b \\ q^{\psi+1}cA(I - Aq)^{-1}b \\ q^{\psi+1}cA^2(I - Aq)^{-1}b \end{bmatrix} + \begin{bmatrix} 0 \\ cbq^{\psi} \\ cAbq^{\psi} + cbq^{\psi-1} \end{bmatrix} \qquad \text{(C.2)}$$
$$= \begin{bmatrix} cq^{\psi+1}(I - Aq)^{-1}b \\ c(q^{\psi+1}A(I - Aq)^{-1} + Iq^{\psi})b \\ c(q^{\psi+1}A^2(I - Aq)^{-1} + Aq^{\psi} + Iq^{\psi-1})b \end{bmatrix}.$$

We check the following expression:

$$q^{\psi+1}A(I - Aq)^{-1} + q^{\psi}I = q^{\psi}(I - Aq)^{-1}. \qquad \text{(C.3)}$$

Multiplying by $(I - Aq)$ on the right, it is easy to see that it holds. Multiplying both parts of (C.3) by q^{-1}, we get with (C.3) that

$$q^{\psi+1}A^2(I - Aq)^{-1} + Aq^{\psi} + Iq^{\psi-1} = q^{\psi-1}(I - Aq)^{-1}. \qquad \text{(C.4)}$$

Taking into account (4.8) and (C.3), (C.4), expression (C.2) becomes

$$\begin{bmatrix} cq^{\psi+1}(I - Aq)^{-1}b \\ c(q^{\psi+1}A(I - Aq)^{-1} + Iq^{\psi})b \\ c(q^{\psi+1}A^2(I - Aq)^{-1} + Aq^{\psi} + Iq^{\psi-1})b \end{bmatrix} = \begin{bmatrix} q^{\psi}c(I - Aq)^{-1}bq \\ q^{\psi-1}c(I - Aq)^{-1}bq \\ q^{\psi-2}c(I - Aq)^{-1}bq \end{bmatrix} = \psi(q)c(I - Aq)^{-1}bq. \qquad \text{(C.5)}$$

Due to (C.5), (C.2), (4.6), expression (C.1) transforms into the following:

$$q^{\psi+1}(I - Aq)^{-1}b = l(q) + (L^0)^{-1}L^0\left(q^{\psi+1}(I - Aq)^{-1}b - l(q)\right)$$
$$= l(q) + (L^0)^{-1}\psi(q)c(I - Aq)^{-1}bq = l(q) + \lambda(q)c(I - Aq)^{-1}bq,$$

so

$$q^{\psi+1}(I - Aq)^{-1}b = l(q) + \lambda(q)c(I - Aq)^{-1}bq. \qquad \text{(C.6)}$$

Finally, with (C.6) we can rewrite (4.15) as

$$\begin{aligned} D^{(1)}(q) &= |I - Aq| \left(\tilde{d}(q) - K^{[1]} l(q) - K^{[1]} \lambda(q) c (I - Aq)^{-1} bq\right) \\ &= |I - Aq| \left(\tilde{d}(q) - K^{[1]} \left(l(q) + \lambda(q) c (I - Aq)^{-1} bq\right)\right) \\ &= |I - Aq| \left(\tilde{d}(q) - K^{[1]} q^{\psi+1} (I - Aq)^{-1} b\right). \end{aligned} \tag{C.7}$$

It is easy to see that (C.7) coincides with (4.14), i.e., $D^{(1)}(q) = D(q)$. This completes the proof for the case $n = 3$.

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